

Large N_c in chiral perturbation theory*

R. Kaiser^a, H. Leutwyler^b

Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland

Received: 14 July 2000 / Published online: 27 October 2000 – © Springer-Verlag 2000

Abstract. The construction of the effective Lagrangian relevant for the mesonic sector of QCD in the large N_c limit meets with a few rather subtle problems. We thoroughly examine these and show that, if the variables of the effective theory are chosen suitably, the known large N_c counting rules of QCD can unambiguously be translated into corresponding counting rules for the effective coupling constants. As an application, we demonstrate that the Kaplan–Manohar transformation is in conflict with these rules and is suppressed to all orders in $1/N_c$. The anomalous dimension of the axial singlet current generates an additional complication: The corresponding external field undergoes nonmultiplicative renormalization. As a consequence, the Wess–Zumino–Witten term, which accounts for the $U(3)_R \times U(3)_L$ anomalies in the framework of the effective theory, contains pieces that depend on the running scale of QCD. The effect only shows up at nonleading order in $1/N_c$, but requires specific unnatural parity contributions in the effective Lagrangian that restore renormalization group invariance.

1 Introduction

The low energy properties of QCD are governed by an approximate, spontaneously broken symmetry, which originates in the fact that three of the quarks happen to be light. If m_u, m_d, m_s are turned off, the symmetry becomes exact. The spectrum of the theory then contains eight massless pseudoscalar mesons, the Goldstone bosons connected with the spontaneous symmetry breakdown.

If the number of colors is taken large, the quark loop graph which gives rise to the $U(1)$ anomaly is suppressed [1]. This implies that, in the limit $N_c \rightarrow \infty$, the singlet axial current is also conserved: The theory in effect acquires a higher degree of symmetry. Since the operator $\bar{q}q$ fails to be invariant under the extra $U(1)$ symmetry, the formation of a quark condensate, $\langle 0|\bar{q}q|0\rangle \neq 0$, implies that this symmetry is also spontaneously broken [2]. The spectrum of QCD, therefore, contains a ninth state, the η' , which becomes massless if not only m_u, m_d, m_s are turned off, but if in addition the number of colors is sent to infinity.

Chiral symmetry imposes strong constraints on the properties of the Goldstone bosons. These may be worked out in a systematic manner by means of the effective Lagrangian method, which describes the low energy structure of the theory in terms of an expansion in powers of energies, momenta and quark masses [3,4]. The fact that, in the large N_c limit, the η' also plays the role of a Goldstone boson implies that the properties of this particle are subject to analogous constraints, which may again be worked out by means of a suitable effective Lagrangian.

The main features of the effective theory relevant for the meson sector of QCD in the large N_c limit were discovered long ago [5]. The systematic analysis in the framework of chiral perturbation theory was taken up in [4], where the Green functions of QCD were studied by means of a simultaneous expansion in powers of momenta, quark masses and $1/N_c$. A considerable amount of work concerning the structure of QCD at large N_c has been carried out since then [6–22] and quite a few phenomenological applications of the $1/N_c$ expansion have appeared in the literature. For a review of these, in particular also for a discussion of the η – η' mixing pattern, we refer to [23]. Moreover, the large N_c properties of the effective theory relevant for the baryons were recently investigated in detail [24].

In the present paper, we examine the foundations of the effective theory in the sector with baryon number $B = 0$ and show that this leads to new insights into the low energy structure of QCD in the large N_c limit. The framework is more complicated than in the case of $N_c = 3$, because an additional low energy scale appears, related to the mass of the η' . In order to firmly establish our claims, we first need to put the effective theory on a solid basis and demonstrate that the effective coupling constants necessarily obey the counting rules indicated in [4] (Sects. 2–12 and Appendix A). Using these, we then show that the transformation introduced by Kaplan and Manohar is in conflict with the large N_c properties of QCD: The parameter λ occurring in the transformation

$$m_u \rightarrow m_u + \lambda m_d m_s \quad (\text{cyclic } u \rightarrow d \rightarrow s \rightarrow u), \quad (1)$$

vanishes to all orders of the $1/N_c$ expansion (Sect. 13). Next, we discuss the consequences of the fact that the di-

* Work supported in part by Schweizerischer Nationalfonds

^a e-mail: kaiser@itp.unibe.ch

^b e-mail: leutwyler@itp.unibe.ch

mension of the singlet axial current is anomalous and determine the corresponding dependence of the effective coupling constants on the running scale of QCD (Sects. 14–16). The matching between the effective theories relevant for a finite and an infinite number of colors is worked out in Sect. 18 and Appendix B. Finally, in Sects. 19–21 and Appendix C, we examine the modifications required to extend the Wess–Zumino–Witten term from $SU(3)_R \times SU(3)_L$ to $U(3)_R \times U(3)_L$.

Some of the results described in the present paper were reported earlier [20]. The application to the masses, decay constants and photonic transitions of the pseudoscalar mesons will be given elsewhere [25]. A further application concerns the low energy properties of the correlation function

$$\begin{aligned} \chi(q^2) &= -i \int dx e^{iq \cdot x} \langle 0 | T \omega(x) \omega(0) | 0 \rangle, \\ \omega &= \frac{1}{16\pi^2} \text{tr}_c G_{\mu\nu} \tilde{G}^{\mu\nu}, \end{aligned} \quad (2)$$

where we have absorbed the coupling constant in the gluon field. In particular, the manner in which the topological susceptibility $\chi(0)$ and the derivative $\chi'(0)$ depend on the light quark masses is quite remarkable [25, 26].

2 Effective action

Our analysis is based on the known large N_c behavior of QCD. We work with the effective action of this theory, which describes the response of the system to the perturbation generated by a set of external fields,

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^0 + \bar{q} \gamma^\mu (v_\mu + \gamma_5 a_\mu) q - \bar{q} (s - i \gamma_5 p) q - \theta \omega(3)$$

The term $\mathcal{L}_{\text{QCD}}^0$ describes the limit where the masses of the three light quarks and the vacuum angle are set to zero. The external fields $v_\mu(x)$, $a_\mu(x)$, $s(x)$, $p(x)$ represent hermitean 3×3 matrices in flavor space. The mass matrix of the three light quarks is contained in the scalar external field $s(x)$. The vacuum angle $\theta(x)$ represents the variable conjugate to the operator $\omega(x)$ specified in (2). In Euclidean space, the integral

$$\nu = \int dx \omega$$

is the winding number of the gluon field, so that $\omega(x)$ may be viewed as the winding number density.

The effective action represents the logarithm of the vacuum-to-vacuum transition amplitude,

$$e^{iS_{\text{eff}}\{v, a, s, p, \theta\}} = \langle 0 \text{ out} | 0 \text{ in} \rangle_{v, a, s, p, \theta}. \quad (4)$$

The coefficients of the expansion of $S_{\text{eff}}\{v, a, s, p, \theta\}$ in the external fields are the connected correlation functions of the vector, axial, scalar and pseudoscalar quark currents and of the winding number density, in the massless theory. Lorentz invariance implies that these can be decomposed

into scalar functions, with coefficients that contain the external momenta and the tensors $g_{\mu\nu}$, $\epsilon_{\mu\nu\rho\sigma}$. In view of the fact that the square of $\epsilon_{\mu\nu\rho\sigma}$ can be expressed in terms of $g_{\mu\nu}$, there are two categories of contributions: The natural parity part of the effective action, which collects the pieces that do not contain the ϵ -tensor, and the unnatural parity part, where this tensor occurs exactly once.

The consequences of the symmetry properties of QCD for the natural parity part of the effective action are remarkably simple: The Ward identities related to the conservation of the vector and axial currents imply that this part of the effective action is invariant under chiral gauge transformations,

$$\begin{aligned} r'_\mu &= V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger, & l'_\mu &= V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger \\ s' + i p' &= V_R (s + i p) V_L^\dagger, \\ \theta' &= \theta + i \ln \det V_R - i \ln \det V_L, \end{aligned} \quad (5)$$

with $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$ and $V_R(x), V_L(x) \in U(3)$.

Chiral $U(1)$ transformations play an essential role for the properties of the theory at large N_c . It is important that our list of external fields includes a source for the singlet axial current: The trace of the matrix a_μ . The divergence of this current contains an anomaly proportional to ω . The above transformation law automatically accounts for this term, through the shift in the vacuum angle that is generated by a $U(1)$ rotation.

The $U(1)$ anomaly is not the only one relevant in our context, but the remaining anomalies of the effective action are independent of the interaction, so that they only affect the unnatural parity part. We first investigate the low energy structure of the natural parity part, which, as stated above, is gauge invariant: $S_{\text{eff}}\{v', a', s', p', \theta'\}_{\text{nP}} = S_{\text{eff}}\{v, a, s, p, \theta\}_{\text{nP}}$. The unnatural parity part – in particular, the Wess–Zumino–Witten term – only contributes at nonleading orders of the low energy expansion and will be discussed in detail later on.

3 QCD at large N_c

The well-known leading logarithmic formula

$$\frac{g^2}{(4\pi)^2} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_{\text{QCD}}^2)}, \quad \beta_0 = \frac{1}{3}(11N_c - 2N_f), \quad (6)$$

implies that the running coupling constant tends to zero when N_c becomes large, $g^2 \sim 1/N_c$. At leading order of the $1/N_c$ expansion, the Green functions are dominated by those graphs that contain the smallest possible number of quark loops. For the correlation functions of the quark currents, graphs with one such loop generate the leading contributions.

Consider the connected correlation function formed with n_j quark currents $j_i = \bar{q} \Gamma_i q$ and n_ω winding number densities

$$G_{n_j n_\omega} = \langle 0 | T j_1(x_1) \cdots j_{n_j}(x_{n_j}) \omega(y_1) \cdots \omega(y_{n_\omega}) | 0 \rangle_c, \quad (7)$$

and denote the fraction due to graphs with ℓ quark loops by $G_{n_j n_\omega}^\ell$. The large N_c counting rules of perturbation theory imply that this quantity represents a term of order

$$G_{n_j n_\omega}^\ell = O(N_c^{2-\ell-n_\omega}), \quad \ell = 0, 1, \dots \quad (8)$$

The leading power is independent of the number n_j of quark currents, but decreases with n_ω . This implies that the dependence of the effective action on the vacuum angle is suppressed. The leading terms in the $1/N_c$ expansion of the effective action may be characterized by the formula

$$S_{\text{eff}} = N_c^2 S_0\{\vartheta\} + N_c S_1\{v, a, s, p, \vartheta\} + S_2\{v, a, s, p, \vartheta\} + \dots, \quad (9)$$

where S_ℓ collects the contributions from graphs with ℓ quark loops and

$$\vartheta \equiv \frac{\theta}{N_c}. \quad (10)$$

Since the external fields v, a, s, p contained in S_1 are attached to one and the same quark loop, the expansion of S_1 in terms of these fields generates expressions that involve a single trace over the flavor indices. Likewise, S_2 represents a sum of contributions containing at most two flavor traces, etc.

The dominating contributions to the effective action arise from those graphs that do not involve quark lines. Their sum represents the effective action of gluodynamics, which only depends on the vacuum angle. The expansion in powers of θ yields the connected correlation functions of the field $\omega(x)$,

$$S_{\text{eff}}^{\text{GD}} = \frac{i}{2!} \int dx_1 dx_2 \theta(x_1) \theta(x_2) \langle 0 | T \omega(x_1) \omega(x_2) | 0 \rangle_{\text{GD}} - \frac{i}{4!} \int dx_1 \dots dx_4 \theta(x_1) \dots \theta(x_4) \times \langle 0 | T \omega(x_1) \dots \omega(x_4) | 0 \rangle_{\text{GD}} + \dots$$

The leading contribution in the $1/N_c$ expansion of the effective action of gluodynamics coincides with the one occurring in QCD,

$$S_{\text{eff}}^{\text{GD}} = N_c^2 S_0\{\vartheta\} + O(N_c). \quad (11)$$

In particular, the topological susceptibility of gluodynamics¹,

$$\tau_{\text{GD}} \equiv \frac{\langle \nu^2 \rangle_{\text{GD}}}{V} = \int dx \langle 0 | T \omega(x) \omega(0) | 0 \rangle_{\text{GD}},$$

represents a term of order 1.

¹ As written, the formula refers to Euclidean space. In Minkowski space, the expression reads

$$\tau_{\text{GD}} = -i \int dx \langle 0 | T \omega(x) \omega(0) | 0 \rangle_{\text{GD}}$$

Since gluodynamics possesses a mass gap (excitation energy of the lightest glueball), the correlation functions of ω decay on a scale of order Λ_{QCD} . We are analyzing the theory for external fields that vary slowly on this scale. In that regime, we may expand the factor $\theta(x_1) \dots \theta(x_n)$ around the point x_1 and express the functional through a derivative expansion:

$$S_0\{\vartheta\} = \int dx \{-e_0(\vartheta) + \partial_\mu \vartheta \partial^\mu \vartheta e_1(\vartheta) + \dots\}, \quad (12)$$

where $e_0(\vartheta), e_1(\vartheta), \dots$ are ordinary functions of a single variable. The quantity $N_c^2 e_0(\vartheta)$ represents the energy density of the vacuum for the case where, in the vicinity of the point under consideration, the vacuum angle is taken constant. The term $N_c^2 \partial_\mu \vartheta \partial^\mu \vartheta e_1(\vartheta)$ describes the change in the energy density that arises if the vacuum angle has a nonvanishing gradient there, etc.

4 Relation between QCD and gluodynamics

The large N_c counting rules imply that the quantities $e_0(\vartheta), e_1(\vartheta), \dots$, which occur in the derivative expansion (12) of the effective action of gluodynamics, depend on the variables θ and N_c only through the ratio $\vartheta = \theta/N_c$. This is puzzling, because the effective action of QCD is periodic in θ with period 2π . The resolution of the paradox is given in [27], where the relation between QCD and gluodynamics is discussed in detail. The paradox is related to the fact that the partition function Z_{GD} involves a sum over all gluon field configurations, while Z_{QCD} only extends over those on which fermions can live. In gluodynamics, the sum over all configurations involves fractional winding numbers – only the quantity $N_c \nu$ must be an integer. In the presence of fermions, however, the boundary conditions imply that ν itself must be an integer. Even if their mass is sent to infinity, the fermions thus exert a restriction on the gluon configurations to be summed over: Only configurations with integer winding number contribute. This implies that, if either the number of colors or the quark masses are sent to infinity, the QCD partition function does not in general tend to the partition function of gluodynamics. Instead, it approaches the one obtained by restricting the sum over all gluon configurations to those with integer winding number, given by the projection

$$\hat{Z}_{\text{GD}}(\theta) = \frac{1}{N_c} \sum_{k=0}^{N_c-1} Z_{\text{GD}}(\theta + 2\pi k),$$

which removes fractional winding numbers. While $Z_{\text{GD}}(\theta)$ is periodic in θ with period $2\pi N_c$, the projection $\hat{Z}_{\text{GD}}(\theta)$ has the 2π periodicity characteristic of $Z_{\text{QCD}}(\theta)$. If θ is taken constant and the four-dimensional volume V is large, the partition function of gluodynamics is dominated by the contribution from the ground state,

$$Z_{\text{GD}}(\theta) = \exp\{-V N_c^2 e_0(\vartheta)\}. \quad (13)$$

For $N_c \rightarrow \infty$, the partition function of QCD thus approaches the expression

$$Z_{\text{QCD}}(\theta) \rightarrow \frac{1}{N_c} \sum_{k=0}^{N_c-1} \exp\{-VN_c^2 e_0(\vartheta_k)\},$$

$$\vartheta_k = \frac{\theta + 2\pi k}{N_c}. \quad (14)$$

The large V limit picks out the term in the sum for which $e_0(\vartheta_k)$ is minimal. This means that, in the large N_c limit, the vacuum energy density of QCD is determined by

$$\bar{e}_0 = \min_k e_0(\vartheta_k).$$

Since a change in θ by 2π is equivalent to a shift in k , the quantity \bar{e}_0 is periodic with period 2π , while e_0 is periodic only with respect to $\theta \rightarrow \theta + 2\pi N_c$.

The distinction between the large N_c limit of QCD and gluodynamics only matters for values of θ outside the interval $-\pi < \theta < \pi$. Within this region, we have $\bar{e}_0 = e_0$. For all other values, \bar{e}_0 represents the periodic continuation from that interval. In our context, only the infinitesimal neighborhood of the point $\theta = 0$ matters – we are using the external field $\theta(x)$ merely as a technical device to analyze the properties of the Green functions for $\theta = 0$. In that context, we do not need to distinguish between e_0 and \bar{e}_0 .

We add a remark concerning the topological susceptibility. As discussed in [9], the correlation function $\langle 0|T\omega(x)\omega(0)|0\rangle$ is too singular for the integral

$$\chi(p^2) = -i \int d^4x e^{ip \cdot x} \langle 0|T\omega(x)\omega(0)|0\rangle$$

to exist, in QCD as well as in gluodynamics. The corresponding dispersive representation contains two subtractions,

$$\chi(p^2) = \chi(0) + p^2 \chi'(0) + \frac{p^4}{\pi} \int_0^\infty \frac{ds}{s^2(s-p^2)} \text{Im}\chi(s).$$

Accordingly, if the susceptibility is defined as an integral over the correlation function, it is an ambiguous notion. We instead identify it with the mean square winding number per unit volume, $\chi(0) = \langle \nu^2 \rangle / V$, which does not suffer from such an ambiguity. In particular, if the large N_c limit is taken at nonzero quark masses, we have²

$$\tau_{\text{QCD}} = \tau_{\text{GD}} + O(N_c^{-1}), \quad \tau_{\text{GD}} = O(1). \quad (15)$$

The ambiguity in the first derivative remains – it reflects the fact that, in the presence of a space-time-dependent vacuum angle, the Lagrangian of QCD must be supplemented with a contact term $\propto D_\mu \theta D^\mu \theta$. A renormalization of the corresponding coupling constant is needed to absorb the quadratic divergence in the correlation function $\langle 0|T\omega(x)\omega(0)|0\rangle$. An analogous term also occurs at the level of the effective theory.

² The equality only holds if $N_c m$ is large compared to the scale of the theory. In the chiral limit, τ_{QCD} vanishes, so that $\tau_{\text{QCD}} \not\rightarrow \tau_{\text{GD}}$

5 Massless quarks

In the present section, we briefly review some of the consequences of the $U(1)$ anomaly, because these play a central role in the analysis of the low energy structure at large N_c . For simplicity, we consider massless quarks, but leave the number of flavors open. We normalize the singlet axial current with

$$A_\mu^0 = \bar{q} \frac{1}{2} \lambda_0 \gamma_\mu \gamma_5 q = \frac{1}{\sqrt{2N_f}} \bar{q} \gamma_\mu \gamma_5 q, \quad (16)$$

where N_f is the number of flavors. The divergence of this current is given by

$$\partial^\mu A_\mu^0 = \sqrt{2N_f} \omega. \quad (17)$$

We write the η' matrix elements in the form

$$\langle 0|A_\mu^0|\eta'\rangle = ip_\mu F_0, \quad \langle 0|\omega|\eta'\rangle = \sqrt{2N_f} \frac{\tau}{F_0}.$$

The relation (17) then shows that the mass of the η' is given by

$$M_{\eta'}^2 = \frac{2N_f \tau}{F_0^2}, \quad (18)$$

and the Ward identities obeyed by the correlation functions of A_μ^0 and ω lead to the representation

$$\begin{aligned} i \int dx e^{ip \cdot x} \langle 0|T A_\mu^0(x) A_\nu^0(0)|0\rangle &= \frac{p_\mu p_\nu F_0^2}{M_{\eta'}^2 - p^2} \\ &+ g_{\mu\nu} F_0^2 + g_{\mu\nu} R_0(p^2) + (p_\mu p_\nu - g_{\mu\nu} p^2) R_1(p^2), \\ \int dx e^{ip \cdot x} \langle 0|T A_\mu^0(x) \omega(0)|0\rangle &= \frac{p_\mu \sqrt{2N_f} \tau}{M_{\eta'}^2 - p^2} \\ &+ \frac{1}{\sqrt{2N_f}} p_\mu R_0(p^2), \\ \frac{1}{i} \int dx e^{ip \cdot x} \langle 0|T \omega(x) \omega(0)|0\rangle &= -\frac{2N_f \tau^2}{F_0^2 (M_{\eta'}^2 - p^2)} \\ &+ \tau - \frac{1}{2N_f} p^2 R_0(p^2). \end{aligned} \quad (19)$$

The large N_c counting rules of Sect. 3 imply that the correlation function of the axial current is a quantity of $O(N_c)$. For the pole term contained therein to be consistent with this, the decay constant F_0 must be of $O(N_c^{1/2})$. Similarly, for the pole term in the correlation function formed with $A_\mu^0(x)$ and $\omega(x)$ not to generate a contribution that diverges for $N_c \rightarrow \infty$, the constant τ must represent a quantity of $O(1)$. In the large N_c limit, the pole term in $\langle 0|T\omega(x)\omega(0)|0\rangle$ therefore disappears, as it should: That function must approach the correlation function of gluodynamics. In this theory, the mass gap persists when $N_c \rightarrow \infty$, so that the expansion in powers of the momentum is an ordinary Taylor series, which starts with the topological susceptibility,

$$\tau = \tau_{\text{GD}} + O(1/N_c), \quad R_0(p^2) = O(1). \quad (20)$$

In QCD, the low energy structure is more intricate, particularly when N_c becomes large. In addition to the $N_f^2 - 1$ Goldstone bosons generated by the spontaneous breakdown of $SU(N_f)_R \times SU(N_f)_L$, the spectrum contains a further state that also becomes massless in the limit $N_c \rightarrow \infty$. At large N_c , the low energy structure of the theory contains a new scale, $M_{\eta'}$, which is independent of the quark masses and of the intrinsic scale of QCD. The magnitude of the pole terms in the above correlation functions, for instance, is sensitive to the relative size of the momentum p compared to this scale.

6 Simultaneous expansion in p and $1/N_c$

In order to analyze the behavior in the region where the momenta are of the size of $M_{\eta'}$ – in particular, on the mass shell of the η' – we need to treat both p and $M_{\eta'}$ as small quantities. This can be done in a controlled manner in the framework of a simultaneous expansion in powers of momenta and $1/N_c$, ordering the series with [14]

$$p = O(\sqrt{\delta}), \quad 1/N_c = O(\delta). \quad (21)$$

In that bookkeeping, the denominator associated with η' exchange represents a small quantity of $O(\delta)$, so that the corresponding low energy singularities are enhanced. In the preceding section, we considered the $1/N_c$ expansion at fixed p – as noted there, the pole term in the correlation function of $\omega(x)$ then represents a subleading contribution. In the simultaneous expansion we are considering now, however, this term is of the same algebraic order as the one from the topological susceptibility of gluodynamics. Moreover, the functions $R_0(p^2)$ and $R_1(p^2)$ are now expanded in powers of p . Since the leading terms are of order $R_0(0) = O(1)$, $R_1(0) = O(N_c) = O(1/\delta)$, the contributions from these functions are suppressed by one power of δ as compared to the pole terms: For massless quarks, the leading term in the δ expansion of the three correlation functions considered in the preceding section is obtained by simply dropping the contributions from $R_0(p^2)$ and $R_1(p^2)$ and is thus fully determined by the low energy constants τ_{GD} and F_0 .

The result may be converted into a simple statement about the effective action. The correlation functions under discussion specify the part of S_{eff} that is bilinear in the external fields $a_\mu^0(x)$ and $\theta(x)$. At leading order in the δ expansion, this part only involves the constants τ_{GD} and F_0 . The explicit expression reads

$$S_{\text{eff}} = \frac{1}{2} \int dx dy f(x) f(y) \Delta_{\eta'}(x - y) + \frac{1}{2} \int dx \{ F_0^2 a_\mu^0(x) a^{0\mu}(x) - \tau_{GD} \theta(x)^2 \} + \dots,$$

where $f(x)$ is a linear combination of the external fields $\partial^\mu a_\mu^0(x)$ and $\theta(x)$, and $\Delta_{\eta'}(x)$ is the propagator of the η' :

$$f(x) = F_0 \partial^\mu a_\mu^0(x) - \sqrt{2N_f} \frac{\tau_{GD}}{F_0} \theta(x),$$

$$\Delta_{\eta'}(x) = \frac{1}{(2\pi)^4} \int dp \frac{e^{-ip \cdot x}}{M_{\eta'}^2 - p^2 - i\epsilon}.$$

The expression represents the classical action of a free scalar field ψ in the presence of external fields. The relevant Lagrangian is given by

$$\mathcal{L}_\psi = \frac{F_0^2}{4N_f} D_\mu \psi D^\mu \psi - \frac{\tau_{GD}}{2} (\psi + \theta)^2,$$

$$D_\mu \psi = \partial_\mu \psi - 2\langle a_\mu \rangle, \quad (22)$$

where $\langle a_\mu \rangle = a_\mu^0 ((1/2)N_f)^{1/2}$ is the trace of the axial external field. If we assign the fields the weight

$$\theta(x) = O(1), \quad a_\mu(x) = O(\sqrt{\delta}),$$

$$\psi(x) = O(1), \quad (23)$$

all of the contributions occurring in this Lagrangian represent terms of $O(1)$.

Above, we determined the effective Lagrangian from the properties of the correlation functions. We could instead have investigated the general expression permitted by the symmetries of the theory – indeed, we will make use of that method to work out the higher order terms in the δ expansion. At leading order, however, that approach leads to the following problem, which arises from the presence of a new low energy scale and does not occur in the effective theory relevant for finite N_c . Let us switch the external fields off. Up to total derivatives, the general Lorentz invariant expression quadratic in ψ is of the form

$$\mathcal{L} = \sum_{n=0}^{\infty} c_n \psi \square^n \psi + \dots$$

A priori, it is not legitimate to dismiss terms with more than two derivatives: If the coefficient c_n is of order N_c^n , then it does contribute at the leading order of the δ expansion. There is a good reason, however, why such terms do not occur. The point is that the propagator of the field ψ is given by the inverse of the function $\sum_n c_n (-p^2)^n$. Unless this expression is a second order polynomial in p , the propagator will thus have more than one zero. With $c_n \propto N_c^n$, all of these occur in the region where p^2 is small, of order $1/N_c$ – the spectrum of the theory would thus contain more than one particle that

- (i) has the quantum numbers of the η' and
- (ii) becomes massless in the large N_c limit. Our explicit calculation of the pole terms in the correlation functions relies on the very plausible assumption that only one such particle occurs – this is why higher derivative terms do not occur at leading order in δ .

7 Effective theory in the U(1) sector

The result obtained in the preceding section shows that – as far as the two-point functions of the operators $A_\mu^0(x)$ and $\omega(x)$ are concerned, and for massless quarks – the leading terms in the simultaneous expansion in powers of momenta and $1/N_c$ can be characterized by means of a remarkably simple effective field theory. It involves a single dynamical variable $\psi(x)$, describing the degrees of

freedom of the η' . Before generalizing this result to the other correlation functions and to nonzero quark masses, we add a few remarks about the structure of the effective Lagrangian obtained above.

First, we note that the Lagrangian (22) is manifestly invariant under local $U(N_f)_R \times U(N_f)_L$ rotations, provided the dynamical variable is transformed according to

$$\psi' = \psi - i \ln \det V_R + i \ln \det V_L. \quad (24)$$

This ensures that the sum $\psi + \theta$ is invariant, so that the same is true of the term proportional to τ_{GD} . Since the trace of a_μ transforms like an Abelian gauge field, the term $D_\mu \psi$ also represents an invariant.

Next, we observe that the choice of the dynamical variable is not unique. The normalization of the field ψ , for instance, can be chosen such that the kinetic term takes the standard form $(1/2)\partial_\mu \psi \partial^\mu \psi$ (the reason for not doing so is that this would ruin the simplicity of the transformation law (24)). More generally, the variable ψ can be replaced by a function thereof, which moreover may involve the derivatives of ψ and the external fields – the result for the action remains the same.

This implies that the effective Lagrangian is not unique. An infinitesimal change of variables generates a term in the Lagrangian that is proportional to the equation of motion. Conversely, contributions proportional to the equation of motion may always be removed from the Lagrangian by performing a suitable transformation of variables. This freedom is characteristic of effective theories, where the dynamical fields represent mere variables of integration in the path integral and are not of physical interest. The transformation $\psi \rightarrow \psi + \kappa \cdot (\psi + \theta)$, for instance, preserves the transformation law (24), but takes the Lagrangian into

$$\begin{aligned} \mathcal{L}_{\psi'} &= \frac{F_0^2}{4N_f} \left\{ (1 + \kappa)^2 D\psi^2 \right. \\ &\quad + 2\kappa(1 + \kappa) D\psi D\theta + \kappa^2 D\theta^2 \left. \right\} \\ &\quad - \frac{\tau_{\text{GD}}}{2} (1 + \kappa)^2 (\psi + \theta)^2, \end{aligned}$$

with $D_\mu \theta = \partial_\mu \theta + 2\langle a_\mu \rangle$. For this choice of variables, the Lagrangian contains additional terms, proportional to $D_\mu \psi D^\mu \theta$ and $D_\mu \theta D^\mu \theta$, respectively. Note also that the coefficient of the one $\propto (\psi + \theta)^2$ is then not given by the topological susceptibility of gluodynamics when N_c becomes large. In the following we stick to the form of the effective Lagrangian in (22).

The low energy analysis relies on the perturbative expansion of the effective theory. At leading order, the effective action is given by the classical action, which collects the contributions from the tree graphs. The quantum fluctuations of the effective field are treated as corrections. In the standard framework, where the η' does not occur among the effective degrees of freedom, the size of the quantum fluctuations is controlled by the external momenta and by the quark masses $m = O(p^2)$: Graphs involving ℓ loops are suppressed by a factor of or-

der $p^{2\ell}$. In the extended framework, however, these fluctuations involve three scales, the external momenta, the quark masses and the mass of the η' . Their magnitude can be controlled algebraically only if $M_{\eta'}$ is also treated as small. For the consistency of the effective framework, it is essential that the number of colors provides us with an algebraic parameter that controls the size of $M_{\eta'} = O(1/N_c^{1/2})$. In fact, we will see that in the simultaneous expansion in powers of momenta, quark masses and $1/N_c$, the quantum fluctuations only start contributing at order δ^2 .

The problems arising if $M_{\eta'}$ is not treated as a small parameter can be seen already at tree level, where the effective action is obtained by evaluating the classical action at its extremum. There, the field $\psi(x)$ obeys the equation of motion, which is of the form $\square \psi + M_{\eta'}^2 \psi = f$. At leading order in an expansion in powers of p at fixed $M_{\eta'}^2$, this equation reduces to $\psi = f/M_{\eta'}^2 + \dots$. The kinetic term then only occurs as a correction. In the loop integrals, the p expansion replaces the propagators by the series $(M_{\eta'}^2 - p^2)^{-1} = 1/M_{\eta'}^2 + \dots$. It is clear that those integrals cannot be analyzed in this manner.

8 Extension to U(3)

We now discuss the extension required to analyze the full effective action and set $N_f = 3$. The dynamical variables of the effective theory must then account for all particles that become massless in the limit $N_c \rightarrow \infty$, $m \rightarrow 0$. In that limit, the symmetry group of the Hamiltonian is $G = U(3)_R \times U(3)_L$. We assume that this symmetry is spontaneously broken and that the ground state yields a nonzero expectation value for the quark condensate $\langle 0 | \bar{q}_L q_R | 0 \rangle$. While the subgroup generated by the vector charges remains intact [28], the $U(1)_A$ rotations generated by the singlet axial charge are spontaneously broken: The operator $\bar{q}_L q_R$ transforms in a nontrivial manner under these. Accordingly, the ground state is symmetric only under the subgroup $H = U(3)_V$. The dynamical variables of the effective theory live on the coset space $G/H = U(3)$, so that we may collect the effective fields in a matrix $U(x) \in U(3)$. The nine parameters of the coset space correspond to the nine massless pseudoscalar fields needed to describe the Goldstone bosons. The extension from the standard framework, where the effective field is an element of $SU(3)$, to the one we are considering here shows up in the phase of the determinant

$$\det U(x) = e^{i\psi(x)}. \quad (25)$$

The field $\psi(x)$ describes the η' . Under the action of the symmetry group, $U(x)$ transforms with

$$U'(x) = V_R(x) U(x) V_L^\dagger(x). \quad (26)$$

The transformation law (24) represents a special case of this formula, obtained by comparing the determinants of the left and right hand sides. In canonical coordinates, the

matrix $U(x)$ is parametrized in terms of nine pseudoscalar fields $\phi^0(x), \dots, \phi^8(x)$:

$$U(x) = \exp i \sum_{k=0}^8 \lambda_k \phi^k(x), \tag{27}$$

where $\lambda_1, \dots, \lambda_8$ are the Gell-Mann matrices and $\lambda_0 = (2/3)^{1/2}$. In these coordinates, the singlet field $\psi(x)$ is represented by $\psi(x) = 6^{1/2} \phi^0(x)$.

The calculation described in Sect. 6 only concerns the part of the effective Lagrangian that governs the dynamics of the η' field $\psi(x)$. The part relevant for the dynamics of the pseudoscalar octet may, however, be worked out in the same manner. If the quark masses are set equal to zero, the eightfold way is an exact symmetry, so that there is no mixing between octet and singlet. The leading term in the δ expansion of the correlation function of the octet components of the axial current, $\langle 0 | T A_\mu^i(x) A_\nu^k(0) | 0 \rangle$, is obtained along the same lines as in the case of the singlet component. The result may again be characterized by an effective Lagrangian that is quadratic in the corresponding dynamical variables – the fields $\phi^k(x)$ in this case:

$$\mathcal{L}_\phi = \frac{1}{2} F^2 \sum_{k=1}^8 (\partial_\mu \phi^k - a_\mu^k) (\partial^\mu \phi^k - a^{\mu k}). \tag{28}$$

In the chiral limit, which we are considering, a mass term does not occur here: The pseudoscalar octet is then strictly massless. The relevant part of the effective Lagrangian contains a single effective coupling constant, related to the matrix element

$$\langle 0 | A_\mu^i | \pi^k \rangle = i p_\mu \delta^{ik} F.$$

In fact, in the large N_c limit, the correlation functions of the singlet and octet currents become identical: The ratio F_0/F tends to 1. Accordingly, the sum $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\phi$ takes the simple form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} F^2 \sum_{k=0}^8 (\partial_\mu \phi^k - a_\mu^k) (\partial^\mu \phi^k - a^{\mu k}) \\ &\quad - \frac{\tau_{\text{GD}}}{2} (\psi + \theta)^2. \end{aligned} \tag{29}$$

9 Chiral symmetry

There is an essential difference between the Lagrangians relevant for the octet and singlet degrees of freedom: While \mathcal{L}_ψ is invariant under chiral transformations, \mathcal{L}_ϕ is not. The difference arises from the fact that the part of the chiral group that matters for \mathcal{L}_ψ is the Abelian $U(1)$ factor, while the one that counts for \mathcal{L}_ϕ is a nonabelian group, $SU(3)_R \times SU(3)_L$. This implies that the pseudoscalar octet mesons necessarily interact among themselves: The Ward identities for the various Green functions intertwine these with one another, so that a nonvanishing two-point function can occur only together with Green functions containing more than two currents. In contrast to the one in

(22), the Lagrangians in (28) and (29) cannot stand by themselves.

The interaction terms required by chiral symmetry are well known. They are generated automatically, if the term $\partial_\mu \phi^k - a_\mu^k$ is replaced by the covariant derivative of the effective field. For the matrix $U(x)$ that includes the η' , the covariant derivative is given by

$$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu). \tag{30}$$

Under chiral rotations, this object transforms in the same manner as the field $U(x)$ itself. The expansion in powers of the meson fields thus starts with

$$D_\mu U = i \sum_{k=0}^8 (\partial_\mu \phi^k - a_\mu^k) \lambda^k + \dots$$

Hence, the Lagrangian (29) represents the quadratic term in the chirally invariant expression (as usual, $\langle A \rangle$ stands for the trace of A)

$$\mathcal{L} = \frac{1}{4} F^2 \langle D_\mu U^\dagger D^\mu U \rangle - \frac{\tau_{\text{GD}}}{2} (\psi + \theta)^2. \tag{31}$$

The above calculation demonstrates that this Lagrangian correctly characterizes the two-point functions formed with the winding number density and the singlet and octet components of the axial current – at leading order of the δ expansion and in the chiral limit. In fact, in this limit, the Lagrangian (31) properly accounts for the leading contributions in the δ expansion of all of the Green functions that can be formed with these operators. For a proof we refer to Appendix A.

We may decompose the effective field into a part that only contains the η' and a part that describes the pseudoscalar octet:

$$U = e^{(i/3)\psi} \hat{U}, \quad \det \hat{U} = 1. \tag{32}$$

It is convenient to define the covariant derivative of $\hat{U} \in SU(3)$ by

$$D_\mu \hat{U} = \partial_\mu \hat{U} - i(\hat{v}_\mu + \hat{a}_\mu) \hat{U} + i\hat{U}(\hat{v}_\mu - \hat{a}_\mu), \tag{33}$$

where \hat{v}_μ, \hat{a}_μ are the traceless parts of v_μ, a_μ . By construction, this quantity obeys $\langle \hat{U}^\dagger D_\mu \hat{U} \rangle = 0$, to be compared with $\langle U^\dagger D_\mu U \rangle = i D_\mu \psi$ for $U \in U(3)$. The relation between the two derivatives,

$$D_\mu U = e^{(i/3)\psi} \left\{ D_\mu \hat{U} + \frac{i}{3} D_\mu \psi \hat{U} \right\},$$

implies the identity

$$\langle D_\mu U^\dagger D^\mu U \rangle = \langle D_\mu \hat{U}^\dagger D^\mu \hat{U} \rangle + \frac{1}{3} D_\mu \psi D^\mu \psi,$$

which splits the above Lagrangian into an $SU(3)$ part that exclusively involves the Goldstone boson octet and a $U(1)$ part that only contains the singlet meson field $\psi(x)$. The singlet axial field and the vacuum angle only occur in the $U(1)$ part, while the vector and axial octets only appear in the $SU(3)$ part – the singlet vector field does not enter at all. For massless quarks, the octet and singlet sectors thus separate like oil and water.

10 Full effective Lagrangian

For nonzero quark masses, the properties of the theory may be analyzed in terms of an expansion in powers of these, so that we are then dealing with a triple expansion in

- (i) the number of derivatives,
- (ii) powers of quark masses and
- (iii) powers of $1/N_c$.

It is convenient to generalize the ordering (21) used for the massless theory by counting the quark masses as quantities of order δ :

$$\partial_\mu = O(\sqrt{\delta}), \quad m = O(\delta), \quad 1/N_c = O(\delta). \quad (34)$$

In this bookkeeping, the vacuum angle and the effective fields $U(x)$, $\psi(x)$ are treated as quantities of order 1, while the external fields $v_\mu(x)$, $a_\mu(x)$, $s(x)$, $p(x)$ count as small perturbations of the same order as the derivatives and the quark masses, respectively:

$$\begin{aligned} (U, \psi, \theta) &= O(1), & (v_\mu, a_\mu) &= O(\sqrt{\delta}), \\ (s, p) &= O(\delta). \end{aligned} \quad (35)$$

Collecting the contributions of order 1, δ , δ^2 , \dots , the full effective Lagrangian takes the form (see Appendix A)

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \dots \quad (36)$$

The individual terms only contain a finite number of effective coupling constants. In particular, the leading term $\mathcal{L}^{(0)} = O(\delta^0)$ exclusively involves the coupling constants F , B and τ . Replacing the external fields $s(x)$ and $p(x)$ by

$$\chi(x) \equiv 2B \{s(x) + ip(x)\}, \quad (37)$$

the explicit expression reads

$$\begin{aligned} \mathcal{L}^{(0)} &= \frac{1}{4}F^2 \langle D_\mu U^\dagger D^\mu U \rangle + \frac{1}{4}F^2 \langle U^\dagger \chi + \chi^\dagger U \rangle \\ &\quad - \frac{1}{2}\tau(\psi + \theta)^2. \end{aligned} \quad (38)$$

Note that $\mathcal{L}^{(0)}$ contains vertices of the type $F^2 \partial \phi \partial \phi \phi^{n-2}$, which represent interactions among the pseudoscalar mesons. The corresponding tree graph contribution to the scattering amplitude describes a collision with altogether n particles in the initial and final states. In the large N_c limit, the contribution disappears, in proportion to $F^{2-n} \propto N_c^{1-n/2}$, in accordance with the general counting of powers for the scattering amplitude given in Appendix A. When N_c is sent to infinity, all of the bound states become free particles of zero width. The larger the number of particles participating in the reaction, the smaller the interaction among these.

In the large N_c limit, diagrams with a single quark loop dominate, so that only contributions with a single trace over flavor matrices occur (OZI rule). This property entails that the two-point function of the singlet current approaches the one of the octet components, in particular

$F_0/F \rightarrow 1$. In general, however, the matrix elements of the singlet and octet components are different, even if N_c is taken large. In particular, the OZI rule does not imply that the four-point function of the singlet axial current approaches the one of the octet components – in fact, it does not: The latter contains a four-fold pole, whose residue is proportional to the scattering amplitude. The term represents a contribution of order N_c , which at low energies is fully determined by the pion decay constant. An analogous contribution to the four-point function of the singlet axial current does not occur, because the scattering amplitude $\eta'\eta' \rightarrow \eta'\eta'$ does not pick up a term proportional to $1/F^2$: Crossing symmetry implies that, for massless quarks, the coefficient is proportional to $s + t + u = 4M_{\eta'}^2 = O(1/N_c)$.

The term $\mathcal{L}^{(1)} = O(\delta)$ contains the contributions of $O(N_c p^4)$, $O(p^2)$ and $O(1/N_c)$. The first may be copied from the SU(3) Lagrangian, simply dropping those coupling constants that violate the OZI rule. Concerning the remaining contributions, the terms that are of the same structure as those in $\mathcal{L}^{(0)}$ may be absorbed in the coupling constants F , B , τ (note that the coefficient of the term proportional to $(\psi + \theta)^2$ then differs from the topological susceptibility of gluodynamics – we need to distinguish the coupling constant τ from τ_{GD}). With a suitable choice of the dynamical variables, the explicit expression for $\mathcal{L}^{(1)}$ can be brought to the form

$$\begin{aligned} \mathcal{L}^{(1)} &= L_2 \langle D_\mu U^\dagger D_\nu U D^\mu U^\dagger D^\nu U \rangle \\ &\quad + (2L_2 + L_3) \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle \\ &\quad + L_5 \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle \\ &\quad + L_8 \langle U^\dagger \chi U^\dagger \chi + \chi^\dagger U \chi^\dagger U \rangle \\ &\quad - iL_9 \langle R_{\mu\nu} D^\mu U D^\nu U^\dagger + L_{\mu\nu} D^\mu U^\dagger D^\nu U \rangle \\ &\quad + L_{10} \langle R_{\mu\nu} U L^{\mu\nu} U^\dagger \rangle + \frac{1}{12}F^2 \Lambda_1 D_\mu \psi D^\mu \psi \\ &\quad - \frac{1}{12}F^2 \Lambda_2 i(\psi + \theta) \langle U^\dagger \chi - \chi^\dagger U \rangle \\ &\quad + \frac{1}{12}H_0 D_\mu \theta D^\mu \theta + H_1 \langle R_{\mu\nu} R^{\mu\nu} + L_{\mu\nu} L^{\mu\nu} \rangle \\ &\quad + H_2 \langle \chi^\dagger \chi \rangle. \end{aligned} \quad (39)$$

The covariant derivatives and the field strength tensors are defined by

$$\begin{aligned} D_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu), \\ D_\mu \psi &= \partial_\mu \psi - 2ia_\mu, \quad D_\mu \theta = \partial_\mu \theta + 2ia_\mu, \\ R_{\mu\nu} &= \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \\ L_{\mu\nu} &= \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu], \end{aligned} \quad (40)$$

where $r_\mu = v_\mu + a_\mu$ and $l_\mu = v_\mu - a_\mu$. The somewhat queer numbering of the coupling constants arises because we retain the notation introduced in [4] to denote the couplings at first nonleading order. The SU(3) Lagrangian given in that reference contains three independent terms with four derivatives, involving the coupling constants L_1 , L_2 and L_3 , respectively. For $N_c \rightarrow \infty$, all of these are of order N_c , but the OZI rule suppresses the combination $2L_1 - L_2$. The

above expression for those terms that involve four derivatives is obtained from the Lagrangian of [4] by replacing the constant L_1 by $(1/2)L_2$. The counting rules imply that the coupling constants $L_2, L_3, L_5, L_8, L_9, L_{10}$ represent quantities of $O(N_c)$, while A_1, A_2 are of $O(1/N_c)$. Concerning the contact terms, H_0 is of order 1, while H_1 and H_2 are of $O(N_c)$.

The number of independent coupling constants entering the low energy representation of the effective action to first nonleading order is roughly the same as for the SU(3) Lagrangian: While the four parameters $L_1 - (1/2)L_2, L_4, L_6, L_7$ are relegated to next-to-next-to leading order, three new coupling constants appear: τ, A_1, A_2 . The framework allows us to also evaluate the correlation functions formed with the operators A_μ^0 and ω , while the Lagrangian of [4] accounts for these only at leading order. A further virtue of a Lagrangian that explicitly includes the degrees of freedom of the η' is that it equips the contributions from η' exchange with the proper denominator and, for instance, distinguishes the factor $1/M_{\eta'}^2$ from $1/(M_{\eta'}^2 - M_\eta^2)$. In view of $M_\eta^2/M_{\eta'}^2 = 0.33$, this distinction is numerically quite significant. On the other hand, the above framework retains only the next-to-leading terms of the $1/N_c$ expansion. In Sects. 17 and 18, we will consider those contributions of next-to-next-to leading order that are needed to fully match the SU(3) and U(3) theories on their common domain of validity.

11 Potentials

At this point, we establish contact with the notation introduced in [4], where the vertices of the effective Lagrangian are ordered in the same fashion as for the case of SU(3), that is by simply counting powers of momenta and quark masses. We refer to that ordering as the p expansion. Ordering the various couplings in this manner, the Lagrangian takes the form

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -V_0 + V_1 \langle D_\mu U^\dagger D^\mu U \rangle + V_2 \langle U^\dagger \chi \rangle + V_2^* \langle \chi^\dagger U \rangle \\ & + V_3 D_\mu \psi D^\mu \psi + V_4 D_\mu \psi D^\mu \theta \\ & + V_5 D_\mu \theta D^\mu \theta + O(p^4). \end{aligned} \quad (41)$$

The leading term V_0 contains those vertices that do not involve derivatives or quark mass factors. Chiral symmetry implies that the matrix U cannot appear without derivatives, but it does not protect the combination $\psi + \theta$ of the singlet field and the vacuum angle. There are vertices without derivatives containing an arbitrary power of this combination. Their collection V_0 represents a function of the variable $\psi + \theta$. The same applies for the coefficients of those contributions that do involve quark mass factors or derivatives. The effective coupling constants are the coefficients $V_{n,k}$ occurring in their expansion in powers of $\psi + \theta$,

$$V_n = \sum_{k=0}^{\infty} V_{n,k} (\psi + \theta)^k.$$

The functions $V_n(\psi + \theta)$ may be viewed as potentials that describe the dynamics in the U(1) sector, with V_0 as the most important one. The counting rules established in Appendix A imply that – for a suitable choice of the dynamical variables – we have

$$\begin{aligned} V_{0,k} &= O(N_c^{2-k}), \quad \{V_{1,k}, V_{2,k}\} = O(N_c^{1-k}), \\ \{V_{3,k}, V_{4,k}, V_{5,k}\} &= O(N_c^{-k}). \end{aligned}$$

Indeed, these rules were written down already in [4]. The detailed analysis described in the present paper merely fills a gap: The argumentation given in that reference is incomplete, because it does not take the ambiguities into account that arise from the freedom in the choice of the dynamical variables. As demonstrated in the Appendix, these ambiguities are in one-to-one correspondence with those occurring in the off-shell extrapolation of the matrix elements. What we have shown now is that there exists a class of such extrapolations, for which the above counting rules for the effective coupling constants strictly follow from those for the correlation functions. We will make use of this result when analyzing the Kaplan–Manohar transformation.

At leading order, the freedom in the choice of the dynamical variables is hidden in the manner in which the field $U(x)$ is parametrized in terms of the variables $\phi^0(x), \dots, \phi^8(x)$ – that is in the choice of the coordinates on the group U(3). As we are expressing the Lagrangian in terms of $U(x)$, it is irrelevant how these coordinates are chosen. At first nonleading order, however, the ambiguities inherent in the choice of the dynamical variables do show up and the Lagrangian takes a unique form only if that freedom is fixed. As discussed in Appendix A, this can be done by eliminating invariants that vanish if the field $U(x)$ obeys the equation of motion associated with $\mathcal{L}^{(0)}$. The representation for $\mathcal{L}^{(1)}$ in (39) fully exploits this freedom. The eleven couplings occurring there are both complete and independent: With a suitable change of variables, all of the invariants that can be built up to and including $O(\delta)$ reduce to a linear combination of those listed and, conversely, the choice made fixes the dynamical variables – up to transformations of $O(\delta^2)$. In the notation introduced above, the choice of variables made in (39) corresponds to $V_4 = 0$, as far as the terms of $O(\delta)$ are concerned. One readily checks that a suitable change of variables of the type $U \rightarrow e^{if} U$, where f only depends on $\psi + \theta$, removes V_4 to all orders and thereby fixes the dynamical variables up to transformations of $O(p^2)$.

At order p^4 , the natural parity part of the effective Lagrangian contains altogether 52 potentials. For an explicit representation, we refer to [15]. The potentials relevant for the unnatural parity part are given explicitly in Sect. 21. In connection with the Kaplan–Manohar transformation, the symmetry breaking terms of order m^2 are of special interest. These are of the same structure as in the case of the SU(3) Lagrangian:

$$\begin{aligned} \mathcal{L}_{\chi^2} = & W_1 \langle U^\dagger \chi U^\dagger \chi \rangle + W_1^* \langle \chi^\dagger U \chi^\dagger U \rangle + W_2 \langle U^\dagger \chi \rangle^2 \\ & + W_2^* \langle \chi^\dagger U \rangle^2 + W_3 \langle U^\dagger \chi \rangle \langle \chi^\dagger U \rangle + W_4 \langle \chi^\dagger \chi \rangle. \end{aligned} \quad (42)$$

The only difference is that the coupling constants L_6, L_7, L_8 and H_2 are replaced by the potentials W_1, \dots, W_4 , which depend on the variable $\psi + \theta$.

In the $1/N_c$ expansion, most of the potentials occurring in \mathcal{L}_2 start contributing only at next-to-next-to leading order: If only terms of order δ are retained, the effective Lagrangian reduces to the explicit expression in (39).

12 Dependence on the vacuum angle θ

Since the dependence of the various quantities on the vacuum angle is rather peculiar, particularly in the large N_c limit, we now briefly discuss this issue. Throughout this section we restrict ourselves to a constant value of the external field $\theta(x) = \theta$, turn the pseudoscalar field $p(x)$ off and identify the scalar one with the quark mass matrix, $s(x) = m$.

The transformation law (5) shows that a suitable global U(1) transformation, for instance $V_R = e^{(i/3)\theta} \mathbf{1}$, $V_L = \mathbf{1}$, removes the vacuum angle. The external vector and axial fields stay put, but the quark mass matrix undergoes a change: $m \rightarrow m_\theta$. The invariance of the effective action thus implies

$$S\{v, a, \theta, m\} = S\{v, a, 0, m_\theta\}, \quad m_\theta = e^{(i/3)\theta} m, \quad (43)$$

the familiar statement that the vacuum angle only enters in combination with the quark mass matrix: Only the phase $\text{argdet} m_\theta = \text{argdet} m + \theta$ matters.

In the effective Lagrangian, however, the vacuum angle does not enter in this combination. Also, according to (26), the effective field undergoes a change under the above transformation: $U \rightarrow e^{(i/3)\theta} U$. We need to be careful when performing finite transformations on the effective field. In the construction of the effective Lagrangian, only the series in powers of both the external fields and the dynamical variables $\phi(x)$ were considered. If we now take a vacuum angle of finite size, we are summing the expansion in powers of θ to all orders. This may lead to ambiguities. The relation $\det U = e^{i\psi}$, for instance, can be solved for ψ only up to a multiple of 2π . This means that the decomposition in (32) is unique only in the vicinity of $U = 1$.

Since the effective theory is evaluated perturbatively, its properties are governed by the tree graphs, that is by those of the corresponding classical field theory. In this framework, ambiguities of the type just mentioned are avoided if ψ is replaced by the gauge invariant field $\bar{\psi} = \psi + \theta$, collecting the remaining dynamical variables in the matrix \bar{U} :

$$U = e^{(i/3)\bar{\psi}} \bar{U}, \quad \bar{\psi} = \psi + \theta, \quad \det \bar{U} = e^{-i\theta}. \quad (44)$$

It is convenient to define the covariant derivative of \bar{U} by

$$D_\mu \bar{U} = \partial_\mu \bar{U} - i(v_\mu + \bar{a}_\mu) \bar{U} + i\bar{U}(v_\mu - \bar{a}_\mu), \quad (45)$$

$$\bar{a}_\mu = a_\mu - \frac{1}{3} \langle a_\mu \rangle - \frac{1}{6} \partial_\mu \theta = a_\mu - \frac{1}{6} D_\mu \theta,$$

so that $\langle \bar{U}^\dagger D_\mu \bar{U} \rangle = 0$. In this notation, the derivative of U reads

$$D_\mu U = e^{(i/3)\bar{\psi}} \left\{ D_\mu \bar{U} + \frac{i}{3} (\partial_\mu \bar{\psi} - D_\mu \theta) \bar{U} \right\}, \quad (46)$$

and the leading term in the effective Lagrangian takes the form

$$\mathcal{L}^{(0)} = \frac{1}{4} F^2 \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} + e^{-(i/3)\bar{\psi}} \bar{U}^\dagger \chi + e^{(i/3)\bar{\psi}} \chi^\dagger \bar{U} \rangle$$

$$+ \frac{1}{12} F^2 (\partial_\mu \bar{\psi} - D_\mu \theta)^2 - \frac{1}{2} \tau \bar{\psi}^2.$$

The equation of motion belonging to this Lagrangian unambiguously determines the classical solution $\bar{\psi}_{\text{cl}}, \bar{U}_{\text{cl}}$ in terms of the external fields. If these are subject to a gauge transformation, $\bar{\psi}_{\text{cl}}$ remains invariant, while \bar{U}_{cl} goes into $V_R \bar{U}_{\text{cl}} V_L^\dagger$, even if the angles of the transformation are taken large. This also holds if the higher order contributions to the effective Lagrangian are accounted for. In particular, under the global U(1) rotation considered above, which maps the vacuum angle into zero, the classical solution does transform with $\bar{U}_{\text{cl}} \rightarrow e^{(i/3)\theta} \bar{U}_{\text{cl}}$.

At leading order of the perturbative expansion, the effective action is given by the value of the classical action at the extremum. Hence it does obey the relation (43), also if θ is taken large. Note that, once the dynamical variables of the effective theory are identified with $\bar{\psi}$ and \bar{U} , the effective Lagrangian contains the external field $\theta(x)$ exclusively through the derivatives thereof – a constant vacuum angle then exclusively manifests itself via the constraint $\det \bar{U} = e^{-i\theta}$. This also implies that the effective theory automatically generates an effective action that is periodic in the vacuum angle: The only place where a constant shift in that angle shows up is through the factor $e^{-i\theta}$, which remains the same if θ is replaced by $\theta + 2\pi$.

The dependence on θ disappears in the large N_c limit, both in gluodynamics and in QCD. The vacuum energy density, for instance, represents a term of $O(N_c^2)$, but a dependence on θ only shows up at $O(1)$, through the susceptibility term. The origin of this suppression can immediately be seen in the structure of the QCD Lagrangian: The properties of the theory are governed by the term $g^{-2} G_{\mu\nu} G^{\mu\nu}$, while the vacuum angle enters through $\theta G_{\mu\nu} \tilde{G}^{\mu\nu}$. In the large N_c limit, the term containing the vacuum angle is smaller than the one that determines the dynamics by the factor $g^2 \theta = O(1/N_c)$.

Gauge invariance implies that, in the effective Lagrangian, the vacuum angle only appears in the combination $\bar{\psi} = \psi + \theta$, together with the singlet field. In particular, the term proportional to θ^2 in the vacuum energy density is converted into one proportional to $\bar{\psi}^2$ and thus equips the η' with a mass. That mass disappears in the large N_c limit, because the θ dependence of the vacuum energy of gluodynamics is suppressed.

Small quark masses also suppress the dependence on θ . If the determinant of the quark mass matrix vanishes, the Green functions of the vector and axial currents and

hence also the scattering matrix elements even become entirely independent of θ . Concerning the θ dependence, the quark masses – the parameters that break chiral symmetry – thus play a similar role as the parameter $1/N_c$ that measures the breaking of the OZI rule.

As an immediate corollary of the fact that, in the large N_c limit, the dependence on the vacuum angle is suppressed, the potentials reduce to constants in that limit. More generally, if the expansion in $1/N_c$ is cut off at a finite order, the potentials are replaced by polynomials in $\bar{\psi}$.

13 Kaplan–Manohar transformation at large N_c

As pointed out by Kaplan and Manohar [29], the standard SU(3) Lagrangian is invariant under the transformation in (1), provided the effective coupling constants L_6, L_7 and L_8 are subject to a corresponding change. In that framework, the vacuum angle is set to zero. The transformation can, however, be generalized to allow for nonzero values of $\theta(x)$. To ensure that the modified mass matrix has the same transformation properties as the original one also with respect to chiral U(1) rotations, it suffices to equip the transformation law for the scalar and pseudoscalar external fields with a factor of $e^{-i\theta}$. With $m(x) = s(x) + ip(x)$, the Kaplan–Manohar transformation then takes the form

$$m \rightarrow m + \lambda e^{-i\theta} m^{\dagger-1} \det m^{\dagger}. \tag{47}$$

In the present context, it is convenient to order the vertices with the p expansion, using the representation of the effective Lagrangian in (41). At order p^2 , the scalar and pseudoscalar external fields only enter through the term $V_2 \langle U^\dagger \chi \rangle$ and its complex conjugate, with $\chi = 2Bm$. Under the above operation, this term picks up a contribution proportional to $\langle U^\dagger \chi^{\dagger-1} \rangle \det \chi^\dagger$. The expression may be simplified by applying the identity

$$\langle C^{-1} \rangle = \frac{\langle C \rangle^2 - \langle C^2 \rangle}{2 \det C}$$

to the matrix $C = \chi^\dagger U$. In view of $\det U = e^{i\psi}$, the transformation then yields

$$V_2 \langle U^\dagger \chi \rangle \rightarrow V_2 \langle U^\dagger \chi \rangle + \frac{\lambda}{4B} V_2 e^{-i(\psi+\theta)} \{ \langle \chi^\dagger U \rangle^2 - \langle \chi^\dagger U \chi^\dagger U \rangle \}.$$

The modification may be absorbed in a suitable change of the potentials that describe the mass terms contained in the effective Lagrangian of order p^4 , which are given in (42). Indeed, the effective Lagrangian does remain invariant under the transformation (47), provided the effective coupling constants contained in the potentials W_1 and W_2 are modified according to

$$\begin{aligned} W_1 &\rightarrow W_1 + \frac{\lambda}{4B} V_2^* e^{i(\psi+\theta)}, \\ W_2 &\rightarrow W_2 - \frac{\lambda}{4B} V_2^* e^{i(\psi+\theta)}, \end{aligned} \tag{48}$$

all other coupling constants remaining as they are. This demonstrates that the general U(3) Lagrangian exhibits the same kind of reparametrization invariance as the one built on SU(3).

The modification of the potentials W_1, W_2 , however, is in conflict with the behavior of the theory at large N_c . As discussed in detail in the preceding section, the effective Lagrangian represents a polynomial in the variable $\psi + \theta$, at any order of the $1/N_c$ expansion – an immediate consequence of the fact that the dependence of the matrix elements on the vacuum angle is suppressed in the large N_c limit. In view of the factor $e^{i(\psi+\theta)}$, the changes required in W_1 and W_2 violate this condition, already at leading order. The disaster evidently originates in the phase factor $e^{-i\theta}$ occurring in the Kaplan–Manohar transformation (47): This factor is needed for the modified quark mass matrix to transform in the same manner as the physical one, but it introduces a dependence on θ that is in conflict with the large N_c properties of the theory.

It does not come as a surprise that the Kaplan–Manohar transformation breaks the Okubo–Iizuka–Zweig rule: The transformation mixes different quark flavors. It is to be expected that the parameter λ can at most be of order $1/N_c$. This also follows from the fact that the coupling constant L_6 picks up a term proportional to λF^2 . Since the counting rules imply that L_6 represents a term of $O(1)$, while F^2 is of order N_c , the transformation can be consistent with the large N_c properties of the theory only if λ represents a term of order $1/N_c$. The above analysis reaches much further: It shows that the parameter λ vanishes to all orders of the $1/N_c$ expansion.

14 Renormalization of the operators A_μ^0 and ω

As is well known, the dimension of the singlet axial current is anomalous [30]. The operator $A_\mu^0 = \bar{q}(1/2)\lambda_0\gamma_\mu\gamma_5q$ must be renormalized for the correlation functions formed with this current to remain finite when the cutoff is removed. In particular, the one particle matrix elements $\langle 0|A_\mu^0|P \rangle = ip_\mu F_P^0$ require renormalization: The singlet decay constants $F_{\pi^0}^0, F_\eta^0, F_{\eta' }^0$ represent quantities like the quark masses or the quark condensate – they must be renormalized.

In the $\overline{\text{MS}}$ scheme, the renormalized quantities depend on a running scale. To distinguish this scale from the one used to renormalize the loop graphs of the effective theory, we denote it by μ_{QCD} . There is a difference in that the scale dependence of the quark masses shows up already at leading order in the $1/N_c$ expansion, while the anomalous dimension of the singlet axial current only manifests itself at next-to-leading order, because the triangle graph responsible for the phenomenon contains an extra quark loop.

The representation of the renormalization group only interrelates operators with the same dimension and the same Lorentz quantum numbers. Moreover, the matrix elements of the representation exclusively involve those

coupling constants that are dimensionless in the dimension of physical interest – the QCD coupling constant g and the vacuum angle θ in the present case. In particular, the representation is independent of the quark masses. If these are turned off, the charges of the flavor group $SU(3)_R \times SU(3)_L$ are conserved. The corresponding representation on the set of field operators commutes with the one of the renormalization group.

Since the axial singlet current does not have any partners with the same Lorentz and flavor quantum numbers and the same dimension, it transforms as a singlet under the renormalization group. The operators $\bar{q}_R^i q_L^k$ form an irreducible representation of the flavor group. As the theory does not contain any other operators with these quantum numbers and with the same dimension, the renormalization is multiplicative also in this case,

$$A_\mu^0 \text{ ren} = Z_A A_\mu^0, \quad (\bar{q}_R^i q_L^k) \text{ ren} = Z_{\bar{q}q} \bar{q}_R^i q_L^k. \quad (49)$$

The renormalization of the singlet decay constants, for instance, reads

$$F_P^0 \text{ ren} = Z_A F_P^0, \quad P = \pi^0, \eta, \eta'.$$

The renormalization factors depend on the running scale of QCD:

$$\begin{aligned} \mu_{\text{QCD}} \frac{dZ_A}{d\mu_{\text{QCD}}} &= \gamma_A Z_A, \\ \gamma_A &= -\frac{6N_f(N_c^2 - 1)}{N_c} \left(\frac{g}{4\pi}\right)^4 + O(g^6), \\ \mu_{\text{QCD}} \frac{dZ_{\bar{q}q}}{d\mu_{\text{QCD}}} &= \gamma_{\bar{q}q} Z_{\bar{q}q}, \\ \gamma_{\bar{q}q} &= \frac{3(N_c^2 - 1)}{N_c} \left(\frac{g}{4\pi}\right)^2 + O(g^4). \end{aligned} \quad (50)$$

The renormalization of the operator ω is more complicated. As this field represents the variable conjugate to θ , the issue is related to the dependence of the effective action on the vacuum angle. This dependence is of crucial importance for the analysis of the theory at large N_c . For this reason, we now discuss it in detail.

The external fields may be viewed as space-time-dependent coupling constants. The renormalization of the operators amounts to a renormalization of these “constants”. In particular, if the vacuum angle is turned off, the effective action remains the same if we replace the bare operators $\bar{q}_R^i q_L^k$, $\bar{q}_L^i q_R^k$ and A_μ^0 by the renormalized ones and at the same time replace the bare external fields by the quantities $s^{\text{ren}} = Z_{\bar{q}q}^{-1} s$, $p^{\text{ren}} = Z_{\bar{q}q}^{-1} p$, $\langle a_\mu \rangle^{\text{ren}} = Z_A^{-1} \langle a_\mu \rangle$, while the vector field and the octet components of the axial field remain put – the corresponding operators do not get renormalized.

To see what happens if the vacuum angle θ is turned on, we exploit the fact that the natural parity part of the effective action is invariant under the local $U(3)_R \times U(3)_L$ transformations specified in (5). We denote the octet part of the axial field by $\hat{a}_\mu \equiv a_\mu - (1/3)\langle a_\mu \rangle$ and replace s , p by the combination

$$m_\theta = e^{(i/3)\theta}(s + ip), \quad (52)$$

which is invariant under the transformations generated by the singlet axial charge. Invariance under this subgroup then implies that $\langle a_\mu \rangle$ and θ can enter the effective action only through the gauge invariant combination $D_\mu \theta = \partial_\mu \theta + 2\langle a_\mu \rangle$,

$$S_{\text{eff}} = S_{\text{eff}}\{v_\mu, \hat{a}_\mu, D_\mu \theta, m_\theta\}. \quad (53)$$

This shows that the functional S_{eff} is known for arbitrary $\theta(x)$ if it is known for $\theta(x) = 0$: The general expression is obtained from the one relevant for $\theta(x) = 0$ by replacing $\langle a_\mu \rangle$, s and p with $(1/2)D_\mu \theta$, $(1/2)(m_\theta + m_\theta^\dagger)$ and $(i/2)(m_\theta^\dagger - m_\theta)$, respectively. In particular, the quantities $D_\mu \theta$ and m_θ are renormalized according to

$$(D_\mu \theta)^{\text{ren}} = Z_A^{-1} D_\mu \theta, \quad m_\theta^{\text{ren}} = Z_{\bar{q}q}^{-1} m_\theta. \quad (54)$$

15 Renormalization of the effective action

For the effective action to remain finite when the cutoff is removed, the Lagrangian must include all terms of mass dimension less than or equal to four that are consistent with the symmetries of the theory. This also applies to contact terms, such as $h_0 D_\mu \theta D^\mu \theta$, $h_1 \langle R_{\mu\nu} R^{\mu\nu} + L_{\mu\nu} L^{\mu\nu} \rangle$ or $h_2 \langle m^\dagger m \rangle$. The constant h_1 , for instance, is needed to renormalize the QCD contributions to the electric charge. It generates contact terms in the correlation functions of the vector and axial currents. Together with the renormalization of the bare coupling constant g , the operation ensures that all of the renormalized correlation functions formed with the quark currents and with ω approach finite limits,

$$\begin{aligned} S_{\text{eff}}\{v_\mu, \hat{a}_\mu, D_\mu \theta, m_\theta, g, h, \mu_0\} = \\ S_{\text{eff}}\{v_\mu, \hat{a}_\mu, (D_\mu \theta)^{\text{ren}}, m_\theta^{\text{ren}}, g^{\text{ren}}, h^{\text{ren}}, \mu_{\text{QCD}}\}. \end{aligned} \quad (55)$$

We have denoted the coupling constants associated with the contact terms collectively by h and use the $\overline{\text{MS}}$ scheme for the renormalized quantities. These are independent of the cutoff μ_0 , but do depend on the running scale μ_{QCD} . In the effective action, the running scale drops out – the left hand side of the above relation is scale independent.

A priori, since θ is dimensionless, the representation of the renormalization group might depend on this coupling constant, too. As discussed above, however, θ is an inessential parameter in the sense that the manner in which the effective action depends on it is fully determined by the symmetries of the theory. In particular, the factors Z_A and $Z_{\bar{q}q}$ are the same as for $\theta = 0$. In fact, the renormalization law for m_θ shows that the vacuum angle is a scale independent quantity – in agreement with the fact that the effective action is periodic in this variable.

The renormalization law for $D_\mu \theta$, on the other hand, implies that the renormalization of the singlet axial field is rather complicated: The effective action approaches a finite limit only if, in addition to a multiplicative renormalization with the factor Z_A^{-1} , the field $\langle a_\mu \rangle$ is simultaneously also subject to a $U(1)$ gauge transformation by the angle $(1/2)(Z_A^{-1} - 1)\theta(x)$,

$$\langle a_\mu \rangle^{\text{ren}} = Z_A^{-1} \langle a_\mu \rangle + \frac{1}{2} (Z_A^{-1} - 1) \partial_\mu \theta. \quad (56)$$

The relation concisely specifies the renormalization of the correlation functions involving the operator ω . In contrast to A_μ^0 and $\bar{q}_R^i q_L^k$, for which the renormalization is multiplicative, ω is subject to an inhomogeneous renormalization, despite the fact that the variable conjugate to it, the vacuum angle, is renormalization group invariant.

The origin of the complication is readily understood: It reflects the conservation law of the singlet axial current

$$\partial^\mu A_\mu^0 = \sigma^0 + \sqrt{6} \omega, \quad \sigma^0 = \sqrt{\frac{2}{3}} \bar{q} i \gamma_5 q \quad (57)$$

(for simplicity, we identify s with the quark mass matrix m and set $p = 0$). While $\partial^\mu A_\mu^0$ picks up a factor of Z_A , the term σ^0 is invariant, because the quark mass matrix transforms contragrediently to the operator $\bar{q} i \gamma_5 q$. The matrix element $\langle 0 | \omega | \eta' \rangle$, for instance, may be represented as a difference of two contributions. The first is proportional to $M_\eta^2 F_\eta^0$ and thus scales with Z_A , while the second is given by the matrix element $\langle 0 | \sigma^0 | \eta' \rangle$, which does not get renormalized.

In contrast to the operator ω itself, the zero momentum projection thereof, the winding number $\nu = \int dx \omega$, is renormalization group invariant. The correlation functions of this quantity may be obtained from the effective action by considering a constant vacuum angle and taking derivatives with respect to it. The term proportional to $\partial_\mu \theta$ occurring in the renormalization of $\langle a_\mu \rangle$ is then absent. Since the vacuum angle is invariant under renormalization, the same holds for the correlation functions of ν .

To illustrate the statement, consider the topological susceptibility of QCD

$$\tau_{\text{QCD}} = -i \int dx \langle 0 | T \omega(x) \omega(0) | 0 \rangle.$$

If s is identified with the quark mass matrix, the vacuum angle is taken constant and all other external fields are switched off, the effective action reduces to $S_{\text{eff}} = - \int dx \epsilon(m, \theta)$, where $\epsilon(m, \theta)$ is the vacuum energy density of QCD. The topological susceptibility is the second derivative of $\epsilon(m, \theta)$ with respect to θ . It can be represented in terms of the correlation function $\langle 0 | T \sigma^0(x) \sigma^0(y) | 0 \rangle$ and the quark condensate, by invoking the Ward identities

$$\begin{aligned} \partial^\mu \langle 0 | T A_\mu^0(x) \omega(y) | 0 \rangle &= \\ \langle 0 | T \sigma^0(x) \omega(y) | 0 \rangle + \sqrt{6} \langle 0 | T \omega(x) \omega(y) | 0 \rangle & \\ \partial^\mu \langle 0 | T A_\mu^0(x) \sigma^0(y) | 0 \rangle &= \\ \langle 0 | T \sigma^0(x) \sigma^0(y) | 0 \rangle + \sqrt{6} \langle 0 | T \omega(x) \sigma^0(y) | 0 \rangle & \\ -i \frac{2}{3} \delta(x - y) \langle 0 | \bar{q} m q | 0 \rangle. & \end{aligned}$$

Since the left hand sides represent total derivatives, they drop out when taking the integral over all of space. This leads to the representation

$$\begin{aligned} \tau_{\text{QCD}} &= \frac{i}{\sqrt{6}} \int dx \langle 0 | T \omega(x) \sigma^0(0) | 0 \rangle \\ &= -\frac{i}{6} \int dx \langle 0 | T \sigma^0(x) \sigma^0(0) | 0 \rangle - \frac{1}{9} \langle 0 | \bar{q} m q | 0 \rangle. \end{aligned}$$

The relation confirms the statement that the susceptibility is renormalization group invariant: Neither the correlation function $\langle 0 | T \sigma^0(x) \sigma^0(0) | 0 \rangle$ nor the term involving the quark condensate, $\langle 0 | \bar{q} m q | 0 \rangle$, depends on the running scale of QCD.

16 Dependence of the effective coupling constants on the running scale of QCD

We now translate these properties of the effective action of QCD into the language of the effective theory. The fact that the operators $\bar{q}_R^i q_L^k$, A_μ^0 and ω must be renormalized implies that some of the effective coupling constants depend on the running scale of QCD. Apart from contact terms, the renormalization is fully determined by the factors Z_A and $Z_{\bar{q}q}$ – the coupling constant g is hidden in the effective couplings. The contact terms h_0, h_1, \dots of the QCD Lagrangian are absorbed in the coupling constants H_0, H_1, \dots of the effective theory. The renormalization of these couplings thus involves the renormalization factors relevant for h_0, h_1, \dots . In the following, we disregard the contact terms altogether – they lead a life of their own. Note that the present section concerns the scale dependence of the effective coupling constants that arises from the renormalization of the QCD Lagrangian. The one generated by the logarithmic divergences that occur within the effective theory is an entirely different issue (see Sect. 17).

If the external fields $\langle a_\mu \rangle$ and θ are switched off, the renormalization exclusively concerns s and p . For the generating functional of the effective theory relevant at fixed N_c to become renormalization group invariant, the low energy constant B must be renormalized with $B^{\text{ren}} = Z_{\bar{q}q} B$. The scale dependence of this constant cancels the one of the fields $s(x)$ and $p(x)$, so that the quantity $\chi = 2B(s + ip)$ is independent of the QCD scale. If the effective SU(3) Lagrangian is written in terms of this variable [4], all of the coupling constants occurring therein, the pion decay constant in particular, are invariant under the renormalization group.

Let us now consider the extension to U(3) and include the external fields $\langle a_\mu \rangle$ and θ , so that the natural parity part of the effective action becomes invariant under local $U(3)_R \times U(3)_L$. The renormalization properties discussed in the preceding section do not rely on the large N_c limit. It is therefore appropriate to return to the form of the effective Lagrangian in (41), which does not invoke the $1/N_c$ expansion. Inserting the decomposition (44), we obtain ($V_4 = 0$):

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= -V_0 + V_1 \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} \rangle + V_2 e^{-(i/3) \bar{\psi}} \langle \bar{U}^\dagger \chi \rangle \\ &+ V_2^* e^{(i/3) \bar{\psi}} \langle \chi^\dagger \bar{U} \rangle + \left(\frac{1}{3} V_1 + V_3 \right) (\partial_\mu \bar{\psi} - D_\mu \theta)^2 \\ &+ V_5 D_\mu \theta D^\mu \theta + O(p^4). \end{aligned} \quad (58)$$

The covariant derivative $D_\mu \bar{U}$ involves the octet components of the vector and axial fields, as well as the vacuum angle. Neither one of these quantities undergoes renormalization, but the singlet axial field in $D_\mu \theta = \partial_\mu \theta + 2\langle a_\mu \rangle$ does: In addition to a multiplicative renormalization, it picks up a U(1) gauge transformation proportional to $\partial_\mu \theta$, so that $(D_\mu \theta)^{\text{ren}} = Z_A^{-1} D_\mu \theta$. For the operation to map the effective Lagrangian onto itself, the variable $\bar{\psi}$ must be renormalized in the same manner:

$$\bar{\psi}^{\text{ren}} = Z_A^{-1} \bar{\psi}, \quad (59)$$

while \bar{U} remains put. The Lagrangian then remains invariant, provided the effective coupling constants are renormalized in such a manner that the potentials transform with

$$\begin{aligned} V_0(x)^{\text{ren}} &= V_0(Z_A x), \\ V_1(x)^{\text{ren}} &= V_1(Z_A x), \\ V_2(x)^{\text{ren}} &= V_2(Z_A x) e^{-(i/3)(Z_A - 1)x}, \\ V_3(x)^{\text{ren}} &= Z_A^2 V_3(Z_A x) + \frac{1}{3}(Z_A^2 - 1)V_1(Z_A x), \\ V_5(x)^{\text{ren}} &= Z_A^2 V_5(Z_A x). \end{aligned} \quad (60)$$

We conclude that the effective Lagrangian is invariant under the renormalization of the external fields specified in Sect. 15, provided the dynamical variable U is subject to renormalization: While the unimodular part remains invariant, the phase $\det U = e^{i\psi}$ transforms with

$$\psi^{\text{ren}} = Z_A^{-1} \psi + (Z_A^{-1} - 1)\theta. \quad (61)$$

Expressed in terms of U , the renormalization thus amounts to

$$U^{\text{ren}} = e^{i(z/3)\theta} (\det U)^{z/3} U, \quad z = Z_A^{-1} - 1. \quad (62)$$

We repeat that we are disregarding contact terms. The above effective Lagrangian does contain such a term: $H_0 = 12V_5(0)$. The renormalization of H_0 involves the one of the QCD counter term h_0 and is not covered by the above renormalization prescription for the potential V_5 . Except for H_0 , the relations (60) do specify the renormalizations of all of the effective coupling constants occurring to order p^2 , in terms of the factors Z_A and Z_{qq} that characterize the anomalous dimensions of the operators A_μ^0 and $\bar{q}_R^i q_L^k$. The couplings collected in the potential V_0 , for instance, are renormalized according to

$$V_{0,k}^{\text{ren}} = Z_A^k V_{0,k}.$$

For the constant $\tau \equiv 2V_{02}$, this yields

$$\tau^{\text{ren}} = Z_A^2 \tau. \quad (63)$$

Note that the factor Z_A differs from 1 only by a term of order $1/N_c$. As the triangle graph responsible for Z_A does not occur in gluodynamics, τ_{GD} does not get renormalized.

The transformation property of $V_1 = (1/4)F^2 + O(x^2)$ shows that F is independent of the QCD scale – in agreement with the fact that F represents the value of a physical quantity in the chiral limit. The constant B transforms

contragrediently to the quark mass matrix, $B^{\text{ren}} = Z_{\bar{q}q} B$, so that the lowest order mass formula $M_\pi^2 = (m_u + m_d)B + \dots$ does yield a scale independent pion mass. For the coupling constants Λ_1 and Λ_2 the above relations yield

$$1 + \Lambda_1^{\text{ren}} = Z_A^2(1 + \Lambda_1), \quad 1 + \Lambda_2^{\text{ren}} = Z_A(1 + \Lambda_2). \quad (64)$$

The result shows that the renormalization of the effective coupling constants is in general not multiplicative. This reflects the fact that the individual terms of the series $\mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \dots$ are not invariant under the renormalization group. The leading term for instance contains the contribution $(1/12)F^2 D_\mu \psi D^\mu \psi$, which picks up a factor of Z_A^{-2} . A term proportional to $D_\mu \psi D^\mu \psi$ also occurs in $\mathcal{L}^{(1)}$. The two contributions add up to $(1/12)F^2(1 + \Lambda_1)D_\mu \psi D^\mu \psi$ – the renormalization of Λ_1 indeed ensures that the sum is renormalization group invariant. Likewise, the renormalization of the potential V_2 generates a term proportional to $Z_A^{-1} - 1$, which is absorbed in the renormalization of Λ_2 . The renormalization group thus intertwines terms in the effective Lagrangian that involve the same number of derivatives and quark mass factors, but carry a different power of N_c . This should barely come as a surprise – the renormalization factor Z_A itself represents an effect of the type $1 + O(1/N_c)$. The renormalization of the nonleading couplings is not multiplicative, because it must cure the deficiencies of the leading terms, so that the results obtained on the basis of the first two terms in the $1/N_c$ expansion of the effective Lagrangian do become renormalization group invariant to first nonleading order.

The remaining coupling constants of $\mathcal{L}^{(1)}$ are renormalization group invariant, because the renormalization of $\mathcal{L}^{(0)}$ does not generate terms of order p^4 . They do, however, give rise to specific nonmultiplicative renormalizations of the couplings occurring in $\mathcal{L}^{(2)}$ (see Sect. 17).

Strictly speaking, the preceding discussion only shows that the renormalization of the effective coupling constants which we have just given is sufficient for the effective action to be independent of the QCD scale. One may also show that this renormalization is necessary – it suffices to calculate a few observables within the effective theory. The scale independence of the result indeed implies that the effective coupling constants must be renormalized in the above fashion [20].

17 Higher orders and loops

There is a significant difference between the U(3) framework considered in the present paper and the standard one, where the degrees of freedom of the meson field are restricted to those of SU(3). In that case, the loop graphs of the effective theory are relevant already at first nonleading order. Now, they only matter if we wish to extend the calculation beyond this order. As far as powers of momenta are concerned, the series is of the same type in the two cases: Momenta count like $p \sim \delta^{1/2}$. The loop graphs, however, are inversely proportional to powers of $F_\pi \sim N_c^{1/2}$. While in the standard chiral perturbation series, graphs containing ℓ loops generate contributions of

order $p^{2\ell}$, they now only manifest themselves at order $\delta^{2\ell}$. In particular, the one loop graphs yield contributions of next-to-next-to leading order.

Numerically, the one loop graphs are of about the same size as those occurring in the chiral perturbation series of SU(3) – to count F_π as a term of order $N_c^{1/2}$ does not change the numerical value of this coupling constant. The one loop graphs are relevant also if we wish to establish the relation between the coupling constants occurring in the two versions of the effective theory: The SU(3) coupling constants depend on the running scale used to renormalize these – we cannot make a significant comparison if we ignore the loops. The evaluation of the one loop graphs, however, poses a problem: Since these violate the OZI rule, the divergences can be absorbed in a renormalization of the effective coupling constants only if we include terms of $O(p^4)$ that are subleading in $1/N_c$.

A complete analysis of the contributions of next-to-next-to leading order is beyond our scope. The corresponding part of the effective Lagrangian contains a plethora of terms of order $N_c p^6$, p^4 , p^2/N_c and $1/N_c^2$, respectively. The first category contains vertices of the same structure as those occurring in the general SU(3) Lagrangian of order p^6 , which are listed in [31]. For a complete set of terms of order p^4 , we refer to [15]. As the one loop graphs of $\mathcal{L}^{(0)}$ represent contributions that involve at most four powers of momentum, their renormalization only requires counter terms that are at most of order p^4 . In other words, the coupling constants of the type $N_c p^6$ are independent of the running scale of the effective theory. Also, since the QCD renormalization group preserves the Lorentz structure of the various vertices, it can intertwine the couplings of $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ only with those terms in $\mathcal{L}^{(2)}$ that are at most of order p^4 . In the following, we restrict ourselves to a small subset of the couplings occurring at next-to-next-to leading order, the minimal set needed to satisfy the following requirements:

- (a) All of the couplings relevant for those quantities that can be calculated within the framework of [4] are included. This allows us to match the two versions of the effective theory at one loop level and to express all of the SU(3) coupling constants in terms of those occurring in the U(3) Lagrangian.
- (b) All contributions needed to absorb the divergences occurring in the one loop graphs for the masses and decay constants of the pseudoscalar nonet are included, so that we can unambiguously account for those contributions of next-to-next-to leading order that are enhanced by a chiral logarithm. The terms required by (a) suffice to satisfy this condition, except for one extra coupling. We retain the numbering introduced in [15] and denote this term by L_{18} .
- (c) The conditions (a) and (b) ensure that the result for the *decay constants* is invariant under the renormalization group of QCD (see Sect. 16). For this to be the case, it is essential that the coupling constant L_{18} is included: The renormalization group intertwines L_5 with this term. Likewise it intertwines L_8 with a further extra coupling constant, L_{25} , which needs to be included for the *masses*

of the pseudoscalars to also become renormalization invariant.

The effective Lagrangian then contains the following couplings:

$$\begin{aligned} \mathcal{L}^{(2)} = & \left(L_1 - \frac{1}{2} L_2 \right) \langle D_\mu U^\dagger D^\mu U \rangle^2 \\ & + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle \\ & + L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 + L_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 \quad (65) \\ & + L_{18} i D_\mu \psi \langle D^\mu U^\dagger \chi - D^\mu U \chi^\dagger \rangle \\ & + L_{25} i (\psi + \theta) \langle U^\dagger \chi U^\dagger \chi - \chi^\dagger U \chi^\dagger U \rangle. \end{aligned}$$

As all of these terms represent contributions of $O(p^4)$, the coefficients approach a finite limit when $N_c \rightarrow \infty$. The contributions involving $L_1 - (1/2)L_2, L_4, L_6$ and L_7 correspond to those pieces of the SU(3) Lagrangian that violate the OZI rule.

The divergences generated by the one loop graphs of $\mathcal{L}^{(0)}$ have been worked out in [15]. Expressed in terms of the factor

$$\lambda = \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left(\ln 4\pi + \Gamma'(1) + 1 \right) \right\}, \quad (66)$$

the renormalization of the effective coupling constants required to absorb these divergences takes the form:

$$\begin{aligned} B &= B^r \left\{ 1 + \frac{4\tau}{F^4} \lambda \right\}, & L_n &= L_n^r + \Gamma_n \lambda, \\ H_n &= H_n^r + \Delta_n \lambda. \end{aligned} \quad (67)$$

The constants F, τ, Λ_1 and Λ_2 do not get renormalized. The main difference to the case of SU(3) is that loops involving the propagation of an η' require a renormalization of the low energy constant B . The same graphs also generate a change in the values of the coefficients Γ_6, Γ_8 and Δ_2 :

$$\begin{aligned} \Gamma_1 &= \frac{3}{32}, & \Gamma_2 &= \frac{3}{16}, & \Gamma_3 &= 0, \\ \Gamma_4 &= \frac{1}{8}, & \Gamma_5 &= \frac{3}{8}, & \Gamma_6 &= \frac{1}{16}, \\ \Gamma_7 &= 0, & \Gamma_8 &= \frac{3}{16}, & \Gamma_9 &= \frac{1}{4}, \\ \Gamma_{10} &= -\frac{1}{4}, & \Gamma_{18} &= -\frac{1}{4}, & \Gamma_{25} &= 0, \\ \Delta_0 &= 0, & \Delta_1 &= -\frac{1}{8}, & \Delta_2 &= \frac{3}{8}. \end{aligned}$$

These relations concern the renormalization of the effective coupling constants required to absorb the divergences occurring within the effective theory – the dependence on the renormalization scale of QCD is a different matter. For the coupling constants of $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$, we have discussed the issue in detail in Sect. 16. That analysis is readily extended to the terms contained in $\mathcal{L}^{(2)}$. We only give the result:

- (a) The coupling constants L_1, L_4, L_6 and L_7 are independent of the QCD scale, also in the framework of U(3).

(b) The constants L_{18} and L_{25} must be renormalized according to

$$\begin{aligned} 2L_5 + 3L_{18}^{\text{ren}} &= Z_A(2L_5 + 3L_{18}), \\ 2L_8 - 3L_{25}^{\text{ren}} &= Z_A(2L_8 - 3L_{25}). \end{aligned} \tag{68}$$

This demonstrates that both L_{18} and L_{25} are needed to arrive at a renormalization group invariant formulation of the effective theory at next-to-next-to leading order.

18 Matching U(3) and SU(3)

The SU(3) framework applies if the quark masses and the external momenta are taken small compared to the mass generated by the topological susceptibility. Quantitatively, the condition takes the form [17]

$$m_s |\langle 0 | \bar{u}u | 0 \rangle| \ll 9\tau_{\text{GD}}. \tag{69}$$

In this region, we may use the straightforward expansion in powers of momenta and quark masses also for the U(3) theory, so that the two effective descriptions have a common region of validity, on which we can compare them.

As explicitly demonstrated in [22], the path integral for the effective U(3) Lagrangian indeed reduces to the one of SU(3) if the singlet meson field is integrated out. The calculation is analogous to the one described in [4], where the SU(3) effective theory is matched with the one relevant for SU(2). Like in that case, loops involving the propagation of light as well as heavy mesons require special attention, because the momentum scale of these is set by the heavy masses, while in the light sector, that scale does not occur. We briefly sketch the essential steps, referring to [22] for a more detailed discussion.

To match the two effective theories, we again use the decomposition introduced in Sect. 12: $U = e^{i(1/3)\bar{\psi}\bar{U}}, \bar{\psi} = \psi + \theta$. The phase factor explicitly exhibits the dependence of U on the singlet field ψ and converts the relation $\det U = e^{i\psi}$ into the constraint $\det \bar{U} = e^{-i\theta}$. The mapping ensures that, under chiral rotations, \bar{U} transforms in the same manner as U . Expressed in terms of these variables, the first term in $\mathcal{L}^{(0)}$ takes the form

$$\begin{aligned} \langle D_\mu U^\dagger D^\mu U \rangle &= \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} \rangle \\ &+ \frac{1}{3} (\partial_\mu \bar{\psi} - D_\mu \theta) (\partial^\mu \bar{\psi} - D^\mu \theta). \end{aligned}$$

In the common domain of validity of the expansions in powers of p and δ , which is characterized by the inequality (69), the equation of motion implies that the field $\bar{\psi}$ represents a term of order p^2 , so that we may expand the expressions in powers of $\bar{\psi}$.

At leading order, the matching reduces to a comparison of the classical actions. Collecting the relevant pieces of $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$, we find that the effective U(3) Lagrangian reduces to

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{1}{4} F^2 \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} \rangle + \frac{1}{4} F^2 \langle \bar{U}^\dagger \chi + \chi^\dagger \bar{U} \rangle \\ &+ \frac{1}{12} \{ H_0 + F^2(1 + \Lambda_1) \} D_\mu \theta D^\mu \theta + O(p^4). \end{aligned}$$

In the notation used here, the leading order SU(3) Lagrangian of [4] reads³

$$\begin{aligned} \mathcal{L}_{p^2}^{\text{SU}_3} &= \frac{1}{4} F^2 \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} \rangle + \frac{1}{4} F^2 \langle \bar{U}^\dagger \chi + \chi^\dagger \bar{U} \rangle \\ &+ \frac{1}{12} (H_0^{\text{SU}_3} + F^2) D_\mu \theta D^\mu \theta. \end{aligned} \tag{70}$$

Hence the two theories match at leading order, provided the coupling constants F and B are the same in both versions and the couplings H_0 are related by

$$H_0^{\text{SU}_3} = H_0 + F^2 \Lambda_1 + O(1/N_c). \tag{71}$$

In order to match the coupling constants of next-to-leading order, we need to perform the integration over the field $\bar{\psi}$, which describes the η' . The key observation here is that $\mathcal{L}^{(0)}$ contains the derivatives of this field exclusively through the term $(1/12)F^2(\partial\bar{\psi} - D\theta)^2$. This implies that – if the vacuum angle and the singlet axial field are turned off – only the vertices proportional to χ generate loops involving the propagation of an η' . Accordingly, the matching of the derivative terms is trivial: The coupling constants $F, L_1, \dots, L_5, L_9, L_{10}$ and H_1 are the same in the two versions of the theory. For B, L_6, L_7, L_8 and H_2 , the matching conditions⁴ read [22]

$$\begin{aligned} B^{\text{SU}_3} &= B \left\{ 1 - \frac{2M_0^2}{3F^2} \lambda_0 + O(N_c^{-3}) \right\}, \\ L_6^{\text{SU}_3} &= L_6 + \frac{1}{72} \left(\lambda_0 - \frac{1}{32\pi^2} \right) + O(N_c^{-1}), \\ L_7^{\text{SU}_3} &= -\frac{F^4(1 + \Lambda_2)^2}{288\tau} + L_7 + O(N_c^{-1}), \\ L_8^{\text{SU}_3} &= L_8 - \frac{1}{12} \lambda_0 + O(N_c^{-1}), \\ H_2^{\text{SU}_3} &= H_2 - \frac{1}{6} \lambda_0 + O(N_c^{-1}), \\ \lambda_0 &= \lambda_{\mu \rightarrow M_0}, \quad M_0^2 = \frac{6\tau}{F^2(1 + \Lambda_1)}. \end{aligned} \tag{72}$$

The quantity M_0 is the mass of the η' in the chiral limit. This mass sets the scale of the logarithm contained in λ_0 . As a check, we note that in the expression for M_0 , the dependence of Λ_1 on the running scale of QCD cancels against the one of τ , in agreement with the fact that the mass of the η' is renormalization group invariant, also in the chiral limit (this is the reason for not expanding the denominator in that expression). The difference between B^{SU_3} and B accounts for the fact that the latter picks up a renormalization from the one loop graphs of the effective theory, while the former does not.

³ In [4], the covariant derivative is defined as $\nabla_\mu \bar{U} = \partial_\mu \bar{U} + i(v_\mu + a_\mu)\bar{U} - i\bar{U}(v_\mu - a_\mu)$. In view of (45), this amounts to $\nabla_\mu \bar{U} = D_\mu \bar{U} - (i/3)D_\mu \theta \bar{U}$

⁴ Note that the low energy constants are independent of the quark masses. The matching conditions do, therefore, not involve M_π, M_K, M_η , but they do contain the mass scale set by the topological susceptibility, which is related to the value of $M_{\eta'}$ in the chiral limit

The difference between the values of L_6, L_8, H_2 in the two versions of the effective theory is related to the difference between the corresponding coefficients $\Gamma_6, \Gamma_8, \Delta_2$. Indeed, one may check that in the matching relations between the renormalized coupling constants, the divergences drop out. The coupling constant L_7 is scale independent in both versions. In the extended theory, L_7 is suppressed by the OZI rule, while $L_7^{\text{SU}_3}$ represents a term of order N_c^2 . The leading contribution arises from η' exchange and is inversely proportional to the square of the mass of the η' [4]:

$$L_7^{\text{SU}_3} \simeq -\frac{F^2}{48M_{\eta'}^2}.$$

According to (72), the first order correction to this formula is determined by the OZI violating coupling constants Λ_1 and Λ_2 :

$$L_7^{\text{SU}_3} = -\frac{F^2(1 + \Lambda_2)^2}{48M_0^2(1 + \Lambda_1)} + L_7 + O(N_c^{-1}). \quad (73)$$

The term L_7 only enters as a correction of second order in $1/N_c$. Note that the dependence on the running scale of QCD also cancels out here: The relation (64) shows that the ratio $(1 + \Lambda_2)^2/(1 + \Lambda_1)$ is renormalization group invariant.

In the present section we considered those couplings of next-to-leading order that survive if θ and a_μ^0 are turned off – the effective Lagrangian of [4] only accounts for these. As shown in Appendix B, the general expression of next-to-leading order involves 11 additional terms. The matching relations for the corresponding coupling constants are also given there.

19 Anomalies

Above, we focussed on the natural parity part of the effective Lagrangian. The photonic decays $\pi^0 \rightarrow \gamma\gamma, \eta \rightarrow \gamma\gamma, \eta' \rightarrow \gamma\gamma$, for instance, are not covered, because these are contained in the unnatural parity part, which collects those terms that involve the tensor $\epsilon^{\mu\nu\rho\sigma}$. We now extend the above discussion to these and first consider the anomalies, which we analyze by means of the differential forms

$$\begin{aligned} v &= dx^\mu v_\mu, & a &= dx^\mu a_\mu, & r &= v + a, \\ l &= v - a, & d &= dx^\mu \partial_\mu. \end{aligned}$$

The quantities dx^0, dx^1, dx^2, dx^3 are treated as Grassmann variables. Their product yields the standard volume element, $dx^\mu dx^\nu dx^\rho dx^\sigma = \epsilon^{\mu\nu\rho\sigma} d^4x$.

The phase of the determinant of the Dirac operator is not invariant under an infinitesimal $U(3)_R \times U(3)_L$ transformation of the external fields,

$$V_R = 1 + i\alpha_R, \quad V_L = 1 + i\alpha_L.$$

The part of the phase change that depends on the gluon field is compensated by the transformation $\delta\theta = \langle \alpha_L - \alpha_R \rangle$

of the vacuum angle. The remainder is unique only up to contact terms formed with the external fields v_μ, a_μ and θ . In the standard convention, it is invariant under the transformations generated by the vector charges, so that the change in the effective action is of the form

$$\delta S_{\text{eff}}\{v, a, s, p, \theta\} = \int \langle (\alpha_L - \alpha_R) \Omega \rangle.$$

An explicit formula⁵ for Ω was given in [32]:

$$\begin{aligned} \Omega &= \frac{N_c}{8\pi^2} \left\{ F_v F_v + \frac{1}{3} D_v a D_v a \right. \\ &\quad \left. + \frac{1}{3} i(F_v a^2 + 4a F_v a + a^2 F_v) + \frac{1}{3} a^4 \right\}, \\ F_v &= dv - iv^2, \quad D_v a = da - iva - iav. \end{aligned} \quad (74)$$

The expression for Ω is not unique, however – in fact, the one given is in conflict with the invariance of the effective action under the renormalization group. The problem arises, because the singlet component of the axial external field transforms in a nontrivial manner under this group. The differential form $a = dx^\mu a_\mu$ consists of two parts (compare Sect. 12),

$$a = \bar{a} + \frac{1}{6} D\theta. \quad (75)$$

The first is renormalization group invariant and transforms as a gauge field under chiral rotations. The second transforms with $(D\theta)^{\text{ren}} = Z_A^{-1} D\theta$ under the renormalization group, but is invariant under $U(3)_R \times U(3)_L$. Inserting the decomposition in the expression for Ω , we obtain a sum of three terms,

$$\Omega = \Omega_0 + \Omega_1 + \Omega_2.$$

The first is obtained from Ω by replacing a with \bar{a} :

$$\begin{aligned} \Omega_0 &= \frac{N_c}{8\pi^2} \left\{ F_v F_v + \frac{1}{3} D_v \bar{a} D_v \bar{a} \right. \\ &\quad \left. + \frac{1}{3} i(F_v \bar{a}^2 + 4\bar{a} F_v \bar{a} + \bar{a}^2 F_v) + \frac{1}{3} \bar{a}^4 \right\}, \end{aligned} \quad (76)$$

where $D_v \bar{a} = d\bar{a} - iv\bar{a} - i\bar{a}v$. It is renormalization group invariant. The remaining two terms are given by

$$\begin{aligned} \Omega_1 &= \frac{N_c}{36\pi^2} \{ D_v \bar{a} \langle da \rangle - i(F_v \bar{a} - \bar{a} F_v) D\theta \}, \\ \Omega_2 &= \frac{N_c}{216\pi^2} \langle da \rangle \langle da \rangle, \end{aligned} \quad (77)$$

and transform with $\Omega_1^{\text{ren}} = Z_A^{-1} \Omega_1, \Omega_2^{\text{ren}} = Z_A^{-2} \Omega_2$. If we were to identify the anomaly of the effective action with Ω , this functional would fail to be independent of the running scale of QCD, in contradiction with (55).

⁵ The sign of Ω is convention dependent; we use the metric $+ - - -$, set $\epsilon_{0123} = +1$ and identify γ_5 with $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$

The problem is readily solved: The extra terms Ω_1 and Ω_2 represent the anomalies generated by two contact terms,

$$P_1 = \frac{N_c}{36\pi^2} \langle \bar{a} D_v \bar{a} \rangle D\theta, \quad P_2 = \frac{N_c}{216\pi^2} \langle a \rangle \langle da \rangle d\theta. \quad (78)$$

Removing these, the anomaly takes the renormalization group invariant form

$$\delta S_{\text{eff}}\{v, a, s, p, \theta\} = \int \langle (\alpha_L - \alpha_R) \Omega_0 \rangle. \quad (79)$$

Since the contact terms P_1, P_2 vanish for $\langle a_\mu \rangle = \partial_\mu \theta = 0$, they only matter when considering correlation functions that contain the operators A_μ^0, ω .

20 Wess–Zumino–Witten term

Within the effective theory, the anomalies are accounted for by the Wess–Zumino–Witten term. The standard expression for this term reads [33]

$$\begin{aligned} S_{\text{WZW}}\{U, v, a\} &= -\frac{iN_c}{240\pi^2} \int_{M_5} \langle \Sigma^5 \rangle \\ &\quad - \frac{iN_c}{48\pi^2} \int_{M_4} \{W(U, r, l) - W(\mathbf{1}, r, l)\}, \\ W(U, r, l) &= \langle UI^3 U^\dagger r + \frac{1}{4} UIU^\dagger r UIU^\dagger r \\ &\quad + iUdlU^\dagger r + idrUIU^\dagger r \\ &\quad - i\Sigma UI^\dagger r UI + \Sigma U^\dagger dr UI \\ &\quad - \Sigma^2 U^\dagger r UI + \Sigma ldl + \Sigma dll \\ &\quad - i\Sigma l^3 + \frac{1}{2} \Sigma l \Sigma l - i\Sigma^3 l \rangle - (R \leftrightarrow L), \end{aligned} \quad (80)$$

with $U \in \text{U}(3)$ and $\Sigma \equiv U^\dagger dU$. The first term is an integral over a field $U(x, x^5)$ that smoothly interpolates between $U(x, 0) = \mathbf{1}$ and $U(x, 1) = U(x)$. The Grassmann algebra is supplemented with a fifth element dx^5 and the integration extends over the five-dimensional manifold M_5 , which represents the direct product of Minkowski space with the interval $0 < x^5 < 1$. The integral is independent of the particular interpolation chosen to connect $U(x)$ with the unit matrix. In the second term, the integration only extends over Minkowski space, M_4 . The operation $(R \leftrightarrow L)$ requires an interchange of the 1-forms r and l as well as an interchange of U and U^\dagger . By construction, an infinitesimal chiral rotation of the variables U, v and a generates the change

$$\delta S_{\text{WZW}}\{U, v, a\} = \int \langle (\alpha_L - \alpha_R) \Omega \rangle, \quad (81)$$

where Ω is the 4-form specified in (74).

For the reason given in the preceding section, we remove the contact terms P_1, P_2 and define the Wess–Zumino–Witten part of the effective Lagrangian through

$$\int dx \mathcal{L}_{\text{WZW}} \equiv S_{\text{WZW}}\{U, v, a\} - \int (P_1 + P_2). \quad (82)$$

This ensures that the term \mathcal{L}_{WZW} accounts for the anomalies of QCD in the renormalization group invariant form (79):

$$\delta \int dx \mathcal{L}_{\text{WZW}} = \int \langle (\alpha_L - \alpha_R) \Omega_0 \rangle. \quad (83)$$

Note, however, that the term \mathcal{L}_{WZW} as such is not renormalization group invariant. Since the scale-dependent pieces contained therein are gauge invariant (see Appendix C), we could remove these and arrive at a scale invariant expression. The drawback of such a choice is that it interferes with the large N_c counting rules: As demonstrated in the next section, our definition of \mathcal{L}_{WZW} is singled out by the property that it represents the leading unnatural parity part of the effective Lagrangian.

We add a remark of general nature. The WZW-Lagrangian is inherently ambiguous: The relation (82) specifies it only up to a total derivative. We may, for instance, indentify \mathcal{L}_{WZW} with what becomes of the right hand side if the integration over Minkowski space is dropped, so that the explicit expression consists of a one-dimensional integral over x^5 and a sum of local terms. The ambiguity then manifests itself in the fact that, in contrast to the action, the Lagrangian does depend on the particular interpolation chosen to connect $U(x, 0) = \mathbf{1}$ with $U(x, 1) = U(x)$. Also, the Lagrangian is invariant under the vectorial subgroup, $V_R(x) = V_L(x)$, only up to a total derivative. Evidently, the properties of \mathcal{L}_{WZW} are less transparent than those of the corresponding action. The path integral relevant for the effective theory, however, only involves the action, so that the ambiguities inherent in \mathcal{L}_{WZW} are irrelevant.

21 Unnatural parity part beyond leading order

We denote the unnatural parity part of the effective Lagrangian by $\tilde{\mathcal{L}}_{\text{eff}}$. As we did not find a complete list for the terms of order p^4 in the literature, we briefly outline the construction.

Once the WZW term is removed, the unnatural parity part also becomes gauge invariant under $\text{U}(3)_R \times \text{U}(3)_L$. It is convenient to express the Lagrangian in terms of the variables $U, \bar{\psi}, v_\mu, a_\mu, D_\mu \theta$ and their derivatives. Terms involving derivatives of $\bar{\psi}$ can be integrated by parts. At order p^4 , charge conjugation invariance then allows six independent invariants ($\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$):

$$\begin{aligned} \tilde{\mathcal{L}}_{\text{eff}} &= \mathcal{L}_{\text{WZW}} + \tilde{V}_1 i \langle \tilde{R}^{\mu\nu} D_\mu U D_\nu U^\dagger + \tilde{L}^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle \\ &\quad + \tilde{V}_2 \langle \tilde{R}^{\mu\nu} U L_{\mu\nu} U^\dagger \rangle + \tilde{V}_3 \langle \tilde{R}^{\mu\nu} R_{\mu\nu} + \tilde{L}^{\mu\nu} L_{\mu\nu} \rangle \\ &\quad + \tilde{V}_4 i D_\mu \theta \langle \tilde{R}^{\mu\nu} D_\nu U U^\dagger - \tilde{L}^{\mu\nu} U^\dagger D_\nu U \rangle \\ &\quad + \tilde{V}_5 (\langle \tilde{R}^{\mu\nu} \rangle \langle R_{\mu\nu} \rangle + \langle \tilde{L}^{\mu\nu} \rangle \langle L_{\mu\nu} \rangle) \\ &\quad + \tilde{V}_6 \langle \tilde{R}^{\mu\nu} \rangle \langle L_{\mu\nu} \rangle + O(p^6). \end{aligned}$$

On account of parity, all of the potentials are odd functions of $\bar{\psi}$, except for \tilde{V}_4 , which is even.

In the $1/N_c$ expansion, the leading contribution to the potentials \tilde{V}_1, \tilde{V}_2 and \tilde{V}_3 is linear in $\bar{\psi}$ and contains a coupling constant of $O(1)$, while \tilde{V}_4 reduces to a constant that

is also of this order. Since the remaining two terms involve two flavor traces, their expansion starts at order $1/N_c$. This implies that the simultaneous expansion in powers of p and $1/N_c$ is dominated by the WZW term, which represents a contributions of order $N_c p^4 = O(\delta)$. The remainder is of order δ^2 or higher:

$$\tilde{\mathcal{L}}_{\text{eff}} = \mathcal{L}_{\text{WZW}} + \tilde{\mathcal{L}}^{(2)} + \tilde{\mathcal{L}}^{(3)} + \dots \quad (84)$$

The general expression for the next-to-leading order Lagrangian contains contributions of the type p^4 and $N_c p^6$. The former can be extracted from the representation given above:

$$\begin{aligned} \tilde{\mathcal{L}}_{p^4}^{(2)} &= \tilde{L}_1 i \bar{\psi} \langle \tilde{R}^{\mu\nu} D_\mu U D_\nu U^\dagger + \tilde{L}^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle \\ &+ \tilde{L}_2 \bar{\psi} \langle \tilde{R}^{\mu\nu} U L_{\mu\nu} U^\dagger \rangle + \tilde{L}_3 \bar{\psi} \langle \tilde{R}^{\mu\nu} R_{\mu\nu} + \tilde{L}^{\mu\nu} L_{\mu\nu} \rangle \\ &+ \tilde{L}_4 i D_\mu \theta \langle \tilde{R}^{\mu\nu} D_\nu U U^\dagger - \tilde{L}^{\mu\nu} U^\dagger D_\nu U \rangle. \end{aligned} \quad (85)$$

At order $N_c p^6$, many invariants can be formed, in particular also terms proportional to the quark mass matrix. Below we will discuss only a selection thereof: The terms relevant in connection with the radiative transitions.

Note that \mathcal{L}_{WZW} is not independent of the scale used to renormalize the singlet axial current. The scale-dependent part is gauge invariant, but represents a contribution of leading order in the $1/N_c$ expansion and must be retained for the relation (84) to hold. As in the case of the natural parity part, renormalization group invariance thus requires specific contributions of nonleading order. The phenomenon arises from the fact that the loop graphs responsible for the anomalous dimension of the singlet current violate the OZI rule, which implies that some of the effective coupling constants contained in $\tilde{\mathcal{L}}^{(2)}$ must compensate for the scale dependence of \mathcal{L}_{WZW} . The relevant terms are those of order p^4 , all of which were listed above. The dependence of the coupling constants on the running scale of QCD follows from the decomposition of the WZW term given in Appendix C: The sum $\mathcal{L}_{\text{WZW}} + \tilde{\mathcal{L}}_{p^4}^{(2)}$ is renormalization group invariant to order δ^2 , provided $\tilde{L}_1, \dots, \tilde{L}_4$ are renormalized according to

$$\begin{aligned} \tilde{L}_1^{\text{ren}} &= Z_A \tilde{L}_1 - \kappa, & \tilde{L}_2^{\text{ren}} &= Z_A \tilde{L}_2 - \kappa, \\ \tilde{L}_3^{\text{ren}} &= Z_A \tilde{L}_3 - \kappa, & \tilde{L}_4^{\text{ren}} &= Z_A \tilde{L}_4 + \kappa, \\ \kappa &= \frac{N_c(Z_A - 1)}{144\pi^2}. \end{aligned} \quad (86)$$

The quantity κ is of $O(1)$, like the coupling constants themselves.

22 Radiative transitions

As an illustration, we consider the radiative transitions $\pi^0 \rightarrow \gamma\gamma$, $\eta \rightarrow \gamma\gamma$, $\eta' \rightarrow \gamma\gamma$ [34]. At leading order, the corresponding part of the effective Lagrangian is obtained from the Wess–Zumino–Witten term with

$$r = l = -eQA,$$

where $Q = \text{diag}\{2/3, -1/3, -1/3\}$ represents the charge matrix of the light quarks and $A = dx^\mu A_\mu$ is the 1-form associated with the electromagnetic field. Setting $U = e^{i\phi}$, the terms linear in ϕ and quadratic in A reduce to

$$\begin{aligned} S_{\text{WZW}}\{U, v, a\} &= \frac{N_c e^2}{8\pi^2} \int \langle Q^2 d\phi \rangle A dA \\ &= -\frac{N_c \alpha}{4\pi} \int d^4x \langle Q^2 \phi \rangle F_{\mu\nu} \tilde{F}^{\mu\nu}. \end{aligned}$$

Concerning $\tilde{\mathcal{L}}_{p^4}^{(2)}$, only the combination $\tilde{L}_2 + 2\tilde{L}_3$ of coupling constants matters for the photonic transition matrix elements. The net effect of the first order OZI violations is that the trace $\langle Q^2 \phi \rangle$ appearing in the WZW term is replaced by

$$\langle Q^2 \phi \rangle \rightarrow \langle Q^2 \phi \rangle + \frac{1}{3} K_1 \langle Q^2 \rangle \langle \phi \rangle,$$

with⁶ $K_1 = -48\pi^2(\tilde{L}_2 + 2\tilde{L}_3)/N_c$. The scaling laws for the coupling constants \tilde{L}_2, \tilde{L}_3 imply that the renormalization of K_1 is of the same form as the one for Λ_2 :

$$1 + K_1^{\text{ren}} = Z_A(1 + K_1). \quad (87)$$

One readily checks that this indeed compensates the renormalization of the singlet field in (59), so that the result for the transition amplitude is renormalization group invariant.

The Lagrangian $\tilde{\mathcal{L}}^{(2)}$ contains two further categories of contributions: Terms of $O(N_c)$ with six derivatives and chiral symmetry breaking effects of $O(N_c m p^4)$. We denote these by $\tilde{\mathcal{L}}_{p^6}^{(2)}$ and $\tilde{\mathcal{L}}_\chi^{(2)}$, respectively. As far as the photonic transitions are concerned, the first contains two independent contributions, which may be written in the form:

$$\tilde{\mathcal{L}}_{p^6}^{(2)} = \tilde{L}_5 e^2 \langle Q^2 \phi \rangle F_{\mu\nu} \square \tilde{F}^{\mu\nu} + \tilde{L}_6 e^2 \langle Q^2 \square \phi \rangle F_{\mu\nu} \tilde{F}^{\mu\nu}.$$

Using the equations of motion, the second term can be absorbed in the couplings occurring in $\tilde{\mathcal{L}}_{p^4}^{(2)}$ and $\tilde{\mathcal{L}}_\chi^{(2)}$. Since the term with $\square F^{\mu\nu}$ only matters for off-shell photons, we can ignore this part of the Lagrangian altogether.

Finally, we consider the chiral symmetry breaking terms. As only neutral mesons are involved, the matrices ϕ, Q and $\chi = 2Bm$ commute, so that there is only one independent invariant with a single flavor trace. We again extract a normalization factor and denote the coupling constant by K_2 :

$$\tilde{\mathcal{L}}_\chi^{(2)} = -\frac{\alpha N_c K_2}{4\pi} \langle Q^2 \chi \phi \rangle F_{\mu\nu} \tilde{F}^{\mu\nu}.$$

The net result for the effective Lagrangian that describes the photonic decays to first nonleading order thus reads:

$$\begin{aligned} \mathcal{L}_{P \rightarrow \gamma\gamma} &= -\frac{\alpha N_c}{4\pi} \left\{ \langle Q^2 \phi \rangle + \frac{1}{3} K_1 \langle Q^2 \rangle \langle \phi \rangle \right. \\ &\quad \left. + K_2 \langle Q^2 \chi \phi \rangle \right\} F_{\mu\nu} \tilde{F}^{\mu\nu}. \end{aligned} \quad (88)$$

⁶ In [17] this coupling constant is denoted by $\Lambda_3 = K_1$

The coupling constant K_1 describes the corrections generated by the violations of the OZI rule and K_2 accounts for the breaking of chiral symmetry. While K_1 represents a term of order $1/N_c$ and depends on the running scale of QCD according to (87), the constant K_2 is of order 1 and is renormalization group invariant.

23 Summary and conclusion

The effective theory of QCD with three colors is well known. In that framework, the effective Lagrangian consists of all terms respecting chiral symmetry. The chiral perturbation series amounts to an expansion in powers of momenta and quark masses. In the present paper, we have examined the extension of this effective theory required to analyze the low energy properties of QCD in the limit where the number of colors, N_c , is treated as large. In that case, the situation is more intricate, because there is an additional low energy scale, related to the mass of the η' . As discussed in detail, the standard expansion in powers of momenta and quark masses, which we refer to as the p expansion, cannot be used when N_c becomes large: In order to coherently analyze the low energy properties of the η' , the two quantities appearing in the η' -propagator – the mass and the square of the momentum – must be treated on equal footing. We exploit the fact that $M_{\eta'}^2$ represents a term of $O(1/N_c)$, so that an expansion that counts p^2 , m and $1/N_c$ as quantities of the same order does have the desired property. We refer to the corresponding low energy expansion as the δ expansion. It orders the triple series in $p = O(\delta^{1/2})$, $m = O(\delta)$ and $1/N_c = O(\delta)$ by collecting terms that are of the same order in δ .

In the limit $N_c \rightarrow \infty$, $g^2 N_c$ fixed, the perturbative analysis of QCD determines the order in $1/N_c$ of the various correlation functions, according to (8). These counting rules only hold for nonexceptional momenta. In particular, the low energy singularities generated by the Goldstone bosons (including the η') give rise to contributions that violate these rules. We have shown, however, that the generating functional $S_{\text{eff}}\{v, a, s, p, \theta\}$ does admit a coherent expansion in powers of δ .

What we then discussed in some detail are the implications of this property of the generating functional for the effective Lagrangian. The reason why this step is not straightforward is closely related to the fact that the dynamical variables of the effective theory do not have physical meaning – their choice is inherently ambiguous. This entails that statements about the effective Lagrangian can only be true modulo a change in the dynamical variables. What we were able to show is that there exists a class of such variables, for which the coupling constants in the effective Lagrangian are at most of $O(N_c)$. This is natural, because the correlation functions collected in the generating functional obey the same bound. Furthermore, we have shown that, like for the generating functional, the dependence on the vacuum angle θ is suppressed in the large N_c limit: At order δ^n , the effective Lagrangian is a polynomial in θ , of degree $n + 2$.

The outcome of our investigation boils down to a remarkably simple construction recipe for the effective Lagrangian that holds to any given order in δ :

(i) Apart from the WZW term, the Lagrangian is manifestly invariant under local $U(3)_R \times U(3)_L$ transformations.
(ii) At order δ^n , the Lagrangian represents a polynomial formed with the fields U , U^\dagger , s , p , $R_{\mu\nu}$, $L_{\mu\nu}$, $\bar{\psi}$, $D_\mu\theta$ and their covariant derivatives. The expression contains terms that are at most of order p^{2n+2} .

(iii) The coupling constant associated with a term in the Lagrangian is of order N_c^{2-k} , where k counts the number of traces plus the number of factors with $\bar{\psi}$, $D_\mu\theta$ or derivatives thereof (note that $k \geq 1$).

In the text we explicitly give the expressions for the Lagrangians of order 1 and order δ , but list only a selection of the vertices occurring at next-to-next-to leading order, for practical reasons: We expect the full Lagrangian of order δ^2 to contain about 100 terms.

We have also shown that the effective theory is consistent with periodicity in the vacuum angle θ . This is not evident, a priori, because the effective Lagrangian truncated at a given order in δ is actually a polynomial in θ . The paradox is resolved by observing that the periodicity is not a necessary feature of the Lagrangian itself – only the corresponding effective action needs to be periodic, and this is the case.

A corollary of our results concerning the structure of the effective Lagrangian is that the transformation of Kaplan and Manohar is forbidden at large N_c : For $\theta \neq 0$, this transformation involves the vacuum angle through a factor $e^{-i\theta}$ and thus generates a modification of the effective Lagrangian with a nonpolynomial dependence on θ . This is in conflict with the properties of the effective theory at large N_c .

As is well known, the singlet axial current A_μ^0 carries anomalous dimension and thus depends on the running scale of QCD. Moreover, the renormalization group mixes the operators ω and $\partial^\mu A_\mu^0$. In the effective action, the corresponding external fields are the trace $\langle a_\mu \rangle$ and the vacuum angle θ . We have shown that their renormalization group properties follow from symmetry considerations alone: While the vacuum angle is scale independent, the singlet field $\langle a_\mu \rangle$ transforms inhomogeneously under scale transformations and picks up a contribution proportional to the gradient of the vacuum angle. We have worked out the consequences for the dynamical variables and coupling constants of the effective theory. In the leading order Lagrangian $\mathcal{L}^{(0)}$, the coupling constant F is scale independent, while B and τ are multiplicatively renormalized. Some of the fields contained therein, however, transform inhomogeneously, so that $\mathcal{L}^{(0)}$ does not remain invariant when the running scale is varied. The origin of the problem – the anomalous dimension of the singlet axial current – is due to graphs of nonleading order. This implies that the change in $\mathcal{L}^{(0)}$ produced by a change of scale is an effect of order $1/N_c$ and is eaten up by a suitable shift of the couplings occurring in $\mathcal{L}^{(1)}$, so that the effective action does remain invariant. Quite generally, the action of the renormalization group on the effective coupling constants

occurring at a given order mixes these with the lower order couplings. Physical quantities only involve scale independent combinations of coupling constants, which are easily identified.

In the remainder of the paper, we have extended the analysis to the unnatural parity part of the Lagrangian and have shown that the scale independence of the effective theory can be made manifest also in this sector. At leading order, the unnatural parity part is given by the Wess–Zumino–Witten term, which accounts for the anomalies within the effective theory. The straightforward extension of this term to the case of $U(3)_R \times U(3)_L$, however, fails to be invariant under the renormalization group. In part, the deficiency only concerns contact contributions that are readily removed. The remainder still contains scale-dependent contributions, but these are gauge invariant, so that the modification produced by a change of the running scale may be absorbed in the coupling constants occurring at the next order of the expansion. The relevant terms are those of unnatural parity at order δ^2 . We could instead have modified the expression for \mathcal{L}_{WZW} , by adding suitable gauge invariant pieces that make it scale independent. This, however, would upset the large N_c counting rules, so that the leading contribution in the δ expansion of the unnatural parity Lagrangian would then not coincide with \mathcal{L}_{WZW} .

To establish contact with the standard low energy theory of QCD, we have given the explicit matching relations between the low energy constants relevant at large and at fixed N_c . We have also discussed the extension needed to investigate singlet currents in the standard framework. Some features, such as the dependence of the effective coupling constants on the running scale of QCD apply to both versions of the theory. Others do not: The Kaplan–Manohar transformation, for instance, is in conflict with the properties of QCD only if the number of colors is treated as large – at fixed N_c , the effective theory is invariant under this operation.

A Construction of the effective Lagrangian

In the present appendix, we show how the reasoning of Sects. 6–9 can be extended to construct the full effective Lagrangian. As input we use the large N_c properties of the Green functions of QCD. We first switch the quark masses off and use the large N_c counting rules for the scattering amplitudes of the pseudoscalar mesons. These allow us to establish corresponding counting rules for the interaction vertices of the effective Lagrangian. Then, we generalize the argument to those vertices that describe the response of the system to the perturbations generated by the external fields and finally discuss the consequences of the Ward identities of chiral symmetry.

Scattering amplitude at large N_c

If the number of colors is sent to infinity and the quark masses are turned off, the spectrum of QCD contains nine

massless pseudoscalar mesons. The scattering amplitudes describing the interaction of any number of these particles in the initial and final state can be extracted from the connected correlation functions formed with the corresponding number of axial currents, using these currents as interpolating fields. The correlation functions represent quantities of order N_c , irrespective of the number of currents contained therein. In view of the fact that the one particle matrix elements of the currents, $\langle 0 | A_\mu^a | \pi^b \rangle$, are of order $N_c^{1/2}$, the scattering amplitude for $n = n_i + n_f$ mesons is at most of order $N_c^{1-n/2}$.

The scattering amplitude contains singularities in the low energy region. In particular, for $n \geq 6$, it contains one particle reducible contributions, describing a sequence of collisions, connected by the exchange of single particles. Denoting the number of exchanged particles by ℓ , there are $\ell + 1$ irreducible parts. The number of meson lines entering or leaving the irreducible parts adds up to $n + 2\ell$. Applying the counting rule to the irreducible parts, the resulting contribution to the scattering amplitude represents a term of order N_c^k , with $k = (\ell + 1) - (1/2)(n + 2\ell) = 1 - (1/2)n$. This shows that the singularities generated by one particle exchange manifest themselves already at leading order.

Unitarity relates the imaginary part to the square of the scattering amplitude. The relation implies that this amplitude contains further singularities. The exchange of a pair of mesons (two particle intermediate states in the unitarity relation), for instance, generates a branch cut in the one particle irreducible parts. The contribution from the two particle cut, however, only shows up at order $N_c^{-n/2}$: If the two parts connected by the two exchanged particles involve n_1 and n_2 mesons, respectively, we have $n_1 + n_2 = n + 4$, so that the overall power of N_c is given by $(1 - (n_1/2)) + (1 - (n_2/2)) = -n/2$. Exchanges of more than two particles between the same irreducible parts are suppressed even more strongly. This means that the branch cuts required by unitarity only show up at non-leading orders of the $1/N_c$ expansion. The leading order contributions only contain those singularities that arise from one particle exchange. Moreover, at leading order, the one particle irreducible parts reduce to polynomials of the momenta.

At leading order of the $1/N_c$ expansion, the structure of the scattering amplitude is the same as the one of the tree graphs of a pseudoscalar field theory. We identify the dynamical variables with the dimensionless fields $\phi^0(x)$, \dots , $\phi^8(x)$ introduced in (27) and represent the interaction Lagrangian in the symbolic form

$$\mathcal{L}_{\text{eff}} = \sum_{k,n} g(k,n) \times \partial^k \times \phi^n, \quad (89)$$

where the flavor and Lorentz structure of the vertices is suppressed. The integers k and n merely count the number of derivatives and fields occurring in the vertex in question and $g(k,n)$ represents the corresponding effective coupling constant – in general, there are several, independent vertices of the same symbolic structure. Lorentz invariance implies that k is even.

The terms quadratic in $\phi(x)$ are given in (22) and (28). The corresponding coupling constants are $g(0, 2) \sim \tau_{\text{GD}} = O(1)$ and $g(2, 2) \sim F^2 = O(N_c)$. For $n > 2$, the coupling constant $g(k, n)$ generates a tree graph contribution to the one particle irreducible scattering amplitude with n mesons. The contribution is of the symbolic form $g(k, n)p^k F^{-n}$, where p stands for the momenta of the particles. The comparison with the counting rule for the scattering amplitude suggests that the coupling constant $g(k, n)$ can at most be of order N_c .

Freedom in the choice of the dynamical variables

Actually, the argument just given runs in the wrong direction: It only shows that if the Lagrangian exclusively contains vertices of order N_c , then the corresponding scattering amplitude does obey the large N_c counting rule – the converse is not true. A counter example can be constructed as follows. As discussed in Sect. 7, the effective Lagrangian is not unique, because its form depends on the choice of variables. We may for instance subject the field $U(x)$ to the transformation $U' = U \exp i f(\psi + \theta)$. The operation preserves the transformation law (24), irrespective of the choice of the function $f(x)$. We may choose one that grows with N_c . Suppose that the coupling constants are of order N_c and express the Lagrangian in terms of the new variables. The resulting expression describes the same physics, but contains effective coupling constants that grow more rapidly than with the first power if N_c becomes large.

The freedom in the choice of the dynamical variables is related to the fact that, in the scattering amplitude, all of the momenta are on the mass shell. It is well known that different vertices may give rise to the same on-shell matrix elements. The requirement that the on-shell matrix elements of a given set of vertices reproduces certain contributions occurring in the scattering amplitude only fixes these matrix elements up to terms that vanish on the mass shell of the colliding particles, $p_i^2 = M_i^2$. In the chiral limit, eight of these are massless, $M_i = 0$, while the mass of the ninth is given by $M_{\eta'}$. The off-shell extension involves an ambiguity of the form $\sum_i (p_i^2 - M_i^2) c_i(p_1, \dots, p_n)$. In coordinate space, this ambiguity corresponds to terms in the Lagrangian that are proportional to the equation of motion, which at leading order in the δ expansion of the massless theory is of the form $\square\phi + \bar{\tau}\langle\phi\rangle = h(\phi)$, with $\bar{\tau} = 2\tau/F^2$. The right hand side, $h(\phi)$, consists of a series of terms that contain three or more fields.

This observation may be used to determine the ambiguity in the effective Lagrangian in an iterative manner. Suppose that the tree graphs of \mathcal{L}_{eff} and $\mathcal{L}'_{\text{eff}}$ generate the same on-shell scattering matrix elements. As discussed in Sect. 7, we may choose the variables such that the terms that are quadratic in the meson fields are the same, so that the difference $\Delta\mathcal{L} = \mathcal{L}'_{\text{eff}} - \mathcal{L}_{\text{eff}}$ only contains vertices with four or more meson fields. Suppose now that the four particle scattering amplitudes generated by these Lagrangians coincide. This property implies that, up to a total derivative, the terms of order ϕ^4 contained in $\Delta\mathcal{L}$ can be written in the form $\sum_k c(k) \times (\square\phi + \bar{\tau}\langle\phi\rangle) \times \partial^k \times \phi^3$.

We may replace $\square\phi + \bar{\tau}\langle\phi\rangle$ with $\square\phi + \bar{\tau}\langle\phi\rangle - h(\phi)$, because the extra contributions contain six or more meson fields. It therefore suffices to transform the variables in $\mathcal{L}'_{\text{eff}}$ with $\phi \rightarrow \phi + F^{-2} \sum_k c(k) \times \partial^k \times \phi^3$. At order ϕ^4 , the operation reduces the difference between the two Lagrangians to a total derivative. Since a change of variables does not modify the physics, the new version of $\mathcal{L}'_{\text{eff}}$ yields the same scattering matrix elements as the original one, irrespective of the number of particles participating in the collision. Iterating the procedure, we may extend this analysis to terms with an arbitrary number of fields. We conclude that the on-shell scattering matrix elements unambiguously determine the effective Lagrangian, except for two inherent degrees of freedom: Choice of the dynamical variables and total derivatives. The first reflects the ambiguities occurring in the extension off the mass shell, the second concerns the extension off the energy-momentum shell.

We will fix the choice of variables when specifying the explicit expressions for the first few terms of the derivative expansion. For the moment, we only exploit the fact that there is a set of coupling constants $g(k, n) = O(N_c)$, for which the tree graphs of the effective Lagrangian do reproduce the scattering amplitudes at leading order of the $1/N_c$ expansion. This only excludes those transformations of variables that generate coupling constants growing more rapidly than with the first power of N_c .

External fields

Let us now turn on the external fields. The above analysis is readily extended to this case. We again consider the massless theory and use the symbol $j_i = \bar{q} \Gamma_i q$ to denote any one of the quark currents. The large N_c counting rules for the correlation functions of these operators were given in Sect. 3. We first generalize this to matrix elements between asymptotic states and consider

$$G_{n_j n_\omega n_i n_f} = \langle f | T j_1(x_1) \cdots j_{n_j}(x_{n_j}) \omega(y_1) \cdots \omega(y_{n_\omega}) | i \rangle_c,$$

where $|i\rangle$ and $\langle f|$ represent states with n_i incoming and n_f outgoing mesons, respectively. The behavior of $G_{n_j n_\omega n_i n_f}$ at large N_c is established in the same manner as for the scattering matrix: The matrix element is related to the residue of the poles occurring in a correlation function that, in addition to the operators listed, involves $n = n_i + n_f$ axial currents, which play the role of the interpolating fields, while the operators $j_1(x_1) \cdots \omega(y_{n_\omega})$ are treated as spectators. Denoting the contribution to $G_{n_j n_\omega n_i n_f}$ that arises from graphs with ℓ quark loops by $G_{n_j n_\omega n_i n_f}^\ell$, the generalization of (8) reads

$$G_{n_j n_\omega n_i n_f}^\ell = O\left(N_c^{2-\ell-n_\omega-(1/2)n_i-(1/2)n_f}\right), \quad \ell = 1, 2, \dots \quad (90)$$

We may also check that, like in the case of the scattering amplitude, the singularities generated by one particle exchange manifest themselves at leading order, while

the unitarity cuts only appear at nonleading orders. In the large N_c limit, the one particle irreducible parts can therefore again be expanded in the momenta.

The above counting rule shows that the matrix elements of all of the quark currents behave in the same manner in the large N_c limit, while those of the operator $\omega(x)$ are suppressed. For the counting of powers relevant at low energies, on the other hand, the external field $\theta(x)$ counts as a term of order 1, $v_\mu(x), a_\mu(x) = O(\sqrt{\delta})$ and $s(x), p(x) = O(\delta)$. In the following bookkeeping, we do not distinguish between $v_\mu(x)$ and $a_\mu(x)$, nor between $s(x)$ and $p(x)$ and write the effective Lagrangian in the symbolic form

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \sum e(k, k_\theta) \times N_c^{2-k_\theta} \times \partial^k \times \theta^{k_\theta} \\ & + \sum g(k, n, k_v, k_s, k_\theta) \times N_c^{1-k_\theta} \\ & \times \partial^k \times \phi^n \times v^{k_v} \times s^{k_s} \times \theta^{k_\theta}. \end{aligned} \quad (91)$$

The first sum accounts for the contributions generated by graphs that do not contain quark lines. It represents the effective Lagrangian of gluodynamics, which we discussed in Sect. 3. The integers k and k_θ count the number of derivatives and the number of times the external field $\theta(x)$ occurs, respectively. In (12), that Lagrangian is written in the form $-N_c^2 e_0(\vartheta) + N_c^2 \partial^\mu \vartheta \partial_\mu e_1(\vartheta) + \dots$, with $\vartheta = \theta/N_c$. As we are now expanding in powers of $1/N_c$, the vacuum energy density $N_c^2 e_0(\vartheta)$ is replaced by the series $N_c^2 e_0(0) + (1/2)\theta^2 e_0''(0) + \dots$ and likewise for the other coefficients. The comparison shows that the coupling constant $e(k, k_\theta)$ is at most of order 1. On account of parity, $e(k, k_\theta)$ vanishes unless k_θ is even. Disregarding the term $e(0, 0) = -N_c^2 e_0(0)$, which merely contributes to the cosmological constant, the sum only starts at $k_\theta = 2$.

The second part of the effective Lagrangian arises from graphs containing at least one quark loop. The integers k and n count the number of derivatives and meson fields, k_v is the number of external vector and axial fields, k_s counts the scalar and pseudoscalar ones, while k_θ is the number of times the field $\theta(x)$ enters. As discussed above in connection with the scattering matrix, the translation of the counting rule for the matrix elements into one for the vertices of the effective Lagrangian involves ambiguities related to the freedom in the choice of the dynamical variables. In the presence of external fields, that freedom becomes even richer, because the variables $\phi^a(x)$ may be subject to transformations that depend on these fields. It suffices to observe, however, that the one particle irreducible matrix elements are polynomials in the momenta, so that their Fourier transforms represent a collection of delta functions and derivatives thereof. We may simply multiply this object with the relevant external fields and add a factor of $F\phi^a(x)$ for each one of the on-shell mesons. Integrating all but one of the coordinates over space, we obtain a specific representation for a term in the effective Lagrangian, for which the relevant tree graph does reproduce the matrix element in question. The effective coupling constants $g(k, n, k_v, k_s, k_\theta)$ occurring therein are the coefficients of the polynomial that describes the matrix element. Their order in the $1/N_c$ expansion is determined

by the counting rule (90), which thus implies that, if the effective Lagrangian is constructed in this manner, the coupling constants $g(k, n, k_v, k_s, k_\theta)$ are at most of order 1. The Lagrangian in (89) is what remains if all external fields are turned off: $g(k, n) \equiv N_c g(k, n, 0, 0, 0)$.

Chiral symmetry

We now consider the simultaneous expansion in powers of momenta (or derivatives) and $1/N_c$, introduced in Sect. 6. The above symbolic expression for the effective Lagrangian explicitly displays the number of derivatives, but only indicates the leading power of N_c : The expansion of the effective coupling constants in powers of $1/N_c$ starts at $O(1)$, but also contains terms of nonleading order. In the following, we stick to this abbreviated notation. With the assignments specified in (34) and (35), the general vertex represents a term of order

$$\begin{aligned} N_c^{1-k_\theta} \times \partial^k \times \phi^n \times v^{k_v} \times s^{k_s} \times \theta^{k_\theta} &= O(\delta^\kappa), \\ \kappa &= \frac{1}{2}(k + k_v) + k_s + k_\theta - 1. \end{aligned}$$

Lorentz invariance implies that $k + k_v$ is even, so that only integer powers of δ occur. Ordering the Lagrangian in this manner, it takes the form

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(-1)} + \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \dots,$$

where the term $\mathcal{L}^{(n)}$ collects all contributions of $O(\delta^n)$. Actually, as shown below, chiral symmetry implies that the first term vanishes. The expansion only begins with the term $\mathcal{L}^{(0)}$, which collects the contributions of $O(1)$. Note that only a finite number of derivatives and external fields can occur at any finite order in δ , so that the relevant Lagrangian only involves a finite number of coupling constants, like in the standard framework. In particular, at $O(\delta^n)$, the dependence on the vacuum angle is a polynomial of degree $n + 2$. This feature reflects the fact that, in the large N_c limit, the θ dependence is suppressed, both in gluodynamics and in massless QCD.

As mentioned in Sect. 2, the Ward identities of chiral symmetry are equivalent to the statement that the effective action is invariant under the $U(3)_R \times U(3)_L$ gauge transformation of the external fields specified in (5). Note that the transformation law only relates quantities of the same order in δ . The effective Lagrangian can therefore be gauge invariant only if this is the case separately for each one of the terms $\mathcal{L}^{(n)}$. In particular, the contributions of order δ^{-1} must altogether represent a gauge invariant expression. Now, these arise from $k = k_v = k_s = k_\theta = 0$ and thus only depend on the meson field. We may think of this part of the effective Lagrangian as being a function of the field $U(x)$, which, moreover, does not involve derivatives thereof, $\mathcal{L}^{(-1)} = f(U)$. Invariance under $U(3)_R \times U(3)_L$ implies that this function obeys $f(V_R U V_L^\dagger) = f(U)$. With $V_L = U, V_R = \mathbf{1}$, this leads to $f(U) = \text{const.}$ Hence $\mathcal{L}^{(-1)}$ only contributes to the cosmological constant and may be discarded.

It is convenient to replace the vacuum angle by $\bar{\psi} = \psi + \theta$ and to use the fields $U, v, a, s, p, \bar{\psi}$ and their derivatives as independent variables – the Lagrangian $\mathcal{L}^{(n)}$ is a gauge invariant function thereof. Moreover, this function is a polynomial in all variables except U . Note that only the fields at one and the same point of space-time enter, so that it is legitimate to treat the derivatives $\partial_\mu U, \partial_\mu \partial_\nu U \dots$ as independent from U . Since $\bar{\psi}$ by itself is invariant under $U(3)_R \times U(3)_L$, gauge invariance does not restrict the dependence on this variable and its derivatives.

At leading order, the above counting rule permits five independent invariants⁷: $\langle U^\dagger D^\mu D_\mu U \rangle$, $\langle D_\mu U^\dagger D^\mu U \rangle$, $\langle (s + ip)U^\dagger \rangle$, $\langle (s - ip)U \rangle$, $\bar{\psi}^2$. The first term differs from the second only by a total derivative and can thus be discarded. For the Lagrangian to be real, the coefficients of $\langle (s + ip)U^\dagger \rangle$ and $\langle (s - ip)U \rangle$ must be complex conjugates of one another and parity then implies that they are real. Hence the leading order Lagrangian contains three independent coupling constants. The explicit expression is given in (38).

It is straightforward to generalize this procedure to find the expressions for the higher order Lagrangians. At order δ , for instance, the relevant terms are those of order p^2 with two flavor traces and those of order p^4 with one trace. Using the equations of motion associated with $\mathcal{L}^{(0)}$, the Lagrangian of order δ may be brought to the form given in (39).

B Full second order SU(3) Lagrangian

In [4], the effective Lagrangian relevant for the low energy analysis of matrix elements involving the winding number density $\omega(x)$ or the singlet vector and axial currents $V_\mu^0(x), A_\mu^0(x)$ was given only at leading order. In the present appendix we briefly discuss the extension needed to study these quantities to first nonleading order of the p expansion. In the language of the effective theory, the relevant extension is obtained by introducing three additional external fields, $\theta(x), \langle v_\mu(x) \rangle, \langle a_\mu(x) \rangle$ and replacing the symmetry group $SU(3)_R \times SU(3)_L$ by $U(3)_R \times U(3)_L$.

The leading order SU(3) Lagrangian is given in (70). At order p^4 , the terms listed in [4] can be taken over as they are, simply replacing the variables U and $\nabla_\mu U$ with the quantities \bar{U} and $D_\mu \bar{U}$, defined in Sect. 12:

$$\begin{aligned} \mathcal{L}_A^{\text{SU}_3} = & L_1^{\text{SU}_3} \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} \rangle^2 \\ & + L_2^{\text{SU}_3} \langle D_\mu \bar{U}^\dagger D_\nu \bar{U} \rangle \langle D^\mu \bar{U}^\dagger D^\nu \bar{U} \rangle \\ & + L_3^{\text{SU}_3} \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} D_\nu \bar{U}^\dagger D^\nu \bar{U} \rangle \\ & + L_4^{\text{SU}_3} \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} \rangle \langle \bar{U}^\dagger \chi + \chi^\dagger \bar{U} \rangle \\ & + L_5^{\text{SU}_3} \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} (\bar{U}^\dagger \chi + \chi^\dagger \bar{U}) \rangle \\ & + L_6^{\text{SU}_3} \langle \bar{U}^\dagger \chi + \chi^\dagger \bar{U} \rangle^2 \\ & + L_7^{\text{SU}_3} \langle \bar{U}^\dagger \chi - \chi^\dagger \bar{U} \rangle^2 \end{aligned}$$

⁷ Note that the term $\langle U^\dagger D^\mu U \rangle \langle U^\dagger D_\mu U \rangle$ involves two flavor traces and thus only occurs at first nonleading order of the expansion

$$\begin{aligned} & + L_8^{\text{SU}_3} \langle \bar{U}^\dagger \chi \bar{U}^\dagger \chi + \chi^\dagger \bar{U} \chi^\dagger \bar{U} \rangle \\ & - i L_9^{\text{SU}_3} \langle \bar{R}_{\mu\nu} D^\mu \bar{U} D^\nu \bar{U}^\dagger + \bar{L}_{\mu\nu} D^\mu \bar{U}^\dagger D^\nu \bar{U} \rangle \\ & + L_{10}^{\text{SU}_3} \langle \bar{R}_{\mu\nu} \bar{U} \bar{L}^{\mu\nu} \bar{U}^\dagger \rangle \\ & + H_1^{\text{SU}_3} \langle \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + \bar{L}_{\mu\nu} \bar{L}^{\mu\nu} \rangle + H_2^{\text{SU}_3} \langle \chi^\dagger \chi \rangle, \end{aligned} \quad (92)$$

where $\bar{R}_{\mu\nu}, \bar{L}_{\mu\nu}$ denote the traceless parts of the right- and lefthanded field strength tensors:

$$\bar{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{3} \langle R_{\mu\nu} \rangle, \quad \bar{L}_{\mu\nu} = L_{\mu\nu} - \frac{1}{3} \langle L_{\mu\nu} \rangle.$$

In addition, chiral symmetry, parity and charge conjugation invariance permit 11 new couplings, containing the covariant derivative of the vacuum angle or the field strengths of the singlet external fields⁸:

$$\begin{aligned} \mathcal{L}_B^{\text{SU}_3} = & -i L_{11}^{\text{SU}_3} D_\mu \theta \langle \bar{U}^\dagger D^\mu \bar{U} D_\nu \bar{U}^\dagger D^\nu \bar{U} \rangle \\ & + L_{12}^{\text{SU}_3} D_\mu \theta D^\mu \theta \langle D_\nu \bar{U}^\dagger D^\nu \bar{U} \rangle \\ & + L_{13}^{\text{SU}_3} D_\mu \theta D_\nu \theta \langle D^\mu \bar{U}^\dagger D^\nu \bar{U} \rangle \\ & + L_{14}^{\text{SU}_3} D_\mu \theta D^\mu \theta \langle \bar{U}^\dagger \chi + \chi^\dagger \bar{U} \rangle \\ & - i L_{15}^{\text{SU}_3} D_\mu \theta \langle D^\mu \bar{U}^\dagger \chi - D^\mu \bar{U} \chi^\dagger \rangle \\ & + i L_{16}^{\text{SU}_3} \partial_\mu D^\mu \theta \langle \bar{U}^\dagger \chi - \chi^\dagger \bar{U} \rangle \\ & + i L_{17}^{\text{SU}_3} \epsilon^{\mu\nu\rho\sigma} D_\mu \theta \langle \bar{R}_{\nu\rho} D_\sigma \bar{U} \bar{U}^\dagger - \bar{L}_{\nu\rho} \bar{U}^\dagger D_\sigma \bar{U} \rangle \\ & + \frac{1}{6} H_3^{\text{SU}_3} \langle R_{\mu\nu} + L_{\mu\nu} \rangle \langle R^{\mu\nu} + L^{\mu\nu} \rangle \\ & + \frac{1}{6} H_4^{\text{SU}_3} \langle R_{\mu\nu} - L_{\mu\nu} \rangle \langle R^{\mu\nu} - L^{\mu\nu} \rangle \\ & + H_5^{\text{SU}_3} (D_\mu \theta D^\mu \theta)^2 + H_6^{\text{SU}_3} (\partial_\mu D^\mu \theta)^2. \end{aligned} \quad (93)$$

Finally, the anomalies of the underlying theory require an extra term that is not gauge invariant, but does not involve any free constants. In the notation introduced in (80), this term is given by

$$\int dx \mathcal{L}_{\text{WZW}}^{\text{SU}_3} \equiv S_{\text{WZW}} \langle \bar{U}, v, \bar{a} \rangle. \quad (94)$$

The full effective Lagrangian of order p^4 reads

$$\mathcal{L}_{p^4}^{\text{SU}_3} = \mathcal{L}_A^{\text{SU}_3} + \mathcal{L}_B^{\text{SU}_3} + \mathcal{L}_{\text{WZW}}^{\text{SU}_3}. \quad (95)$$

Formally, the extension of the low energy analysis to the matrix elements of the operators $\omega(x), A_\mu^0(x), V_\mu^0(x)$ thus nearly doubles the number of effective coupling constants. Four of these, however, represent contact terms and a fifth only concerns matrix elements of unnatural parity (note that a coupling of this type does not occur in the Lagrangian of [4] – $L_{17}^{\text{SU}_3}$ only matters for matrix elements that involve the winding number density or the singlet axial current).

The inclusion of the scale-dependent field $\langle a_\mu \rangle$ implies that some of the coupling constants must be renormalized also in this version of the theory. The renormalization procedure does, however, not entangle coupling constants occurring at different orders in the expansion. In

⁸ In [35], only the coupling constants relevant for the two-point functions were considered. For a corresponding list of U(3) invariants, see [15]

fact, in the above basis, the renormalization is homogeneous: In contrast to a_μ , the field \bar{a}_μ is renormalization group invariant. The constants $L_1^{\text{SU}_3}, \dots, L_{10}^{\text{SU}_3}$ are therefore scale independent, while the remaining couplings pick up a multiplicative renormalization that compensates the one of the external fields,

$$\begin{aligned} (D_\mu \theta)^{\text{ren}} &= Z_A^{-1} D_\mu \theta, \\ \langle R_{\mu\nu}^{\text{ren}} + L_{\mu\nu}^{\text{ren}} \rangle &= \langle R_{\mu\nu} + L_{\mu\nu} \rangle, \\ \langle R_{\mu\nu}^{\text{ren}} - L_{\mu\nu}^{\text{ren}} \rangle &= Z_A^{-1} \langle R_{\mu\nu} - L_{\mu\nu} \rangle. \end{aligned}$$

The renormalizations needed to absorb the infinities generated by the one loop graphs of $\mathcal{L}_{p^2}^{\text{SU}_3}$ may be worked out as follows. The change of variables $\hat{U} = e^{(i/3)\theta} \bar{U}$ takes that Lagrangian into

$$\begin{aligned} \mathcal{L}_{p^2}^{\text{SU}_3} &= \frac{1}{4} F^2 \langle D_\mu \hat{U}^\dagger D^\mu \hat{U} + \hat{U}^\dagger \chi_\theta + \chi_\theta^\dagger \hat{U} \rangle \\ &\quad + \frac{1}{12} \hat{H}_0 D_\mu \theta D^\mu \theta, \\ \det \hat{U} &= 1, \\ D_\mu \hat{U} &= \partial_\mu \hat{U} - i(\hat{v}_\mu + \hat{a}_\mu) \hat{U} + i\hat{U}(\hat{v}_\mu - \hat{a}_\mu), \end{aligned}$$

where \hat{v}_μ and \hat{a}_μ are the traceless parts of the external fields, $\chi_\theta = e^{(i/3)\theta} \chi$ and $\hat{H}_0 = H_0^{\text{SU}_3} + F^2$. In this form, the angle θ and the singlet axial field exclusively occur in χ_θ and $D_\mu \theta$, so that the one loop calculation of [4] can be taken over as it is, simply replacing χ by χ_θ . In particular, that calculation shows that divergences proportional to the new terms do not occur, so that the corresponding renormalization coefficients $\Gamma_{11}, \dots, \Gamma_{17}$ and $\Delta_3, \dots, \Delta_6$ all vanish.

The matching relations for the standard part $\mathcal{L}_A^{\text{SU}_3}$ of the SU(3) Lagrangian are given in Sect. 18. These specify the leading terms in the $1/N_c$ expansion of the coupling constants $L_1^{\text{SU}_3}, \dots, L_{10}^{\text{SU}_3}$. For those in the part involving the singlet external fields, the analogous relations read⁹

$$\begin{aligned} L_{11}^{\text{SU}_3} &= -4 \left(L_2 + \frac{1}{3} L_3 \right) + O(1), \\ L_{12}^{\text{SU}_3} &= \frac{2}{3} \left(L_1 + \frac{1}{2} L_2 + \frac{1}{3} L_3 \right) + O(1), \\ L_{13}^{\text{SU}_3} &= \frac{4}{3} \left(L_2 + \frac{1}{3} L_3 \right) + O(1), \\ L_{14}^{\text{SU}_3} &= \frac{1}{3} (L_4 + 3L_5 + L_{18}) + O(1), \\ L_{15}^{\text{SU}_3} &= \frac{1}{3} (2L_5 + 3L_{18}) + O(1), \\ L_{16}^{\text{SU}_3} &= -F^4 (1 + A_1) (1 + A_2) (72\tau)^{-1} + O(1), \\ L_{17}^{\text{SU}_3} &= N_c (288\pi^2)^{-1} + \frac{1}{2} \tilde{L}_4 + O(N_c^{-1}), \end{aligned}$$

⁹ Note that the independence with respect to changes in the chiral renormalization scale is not manifest in these equations – the dependence of the coupling constants on the scale μ is an effect of order N_c^0 and thus beyond the given accuracy

$$\begin{aligned} H_3^{\text{SU}_3} &= H_1 + \frac{1}{2} L_{10} + O(1), \\ H_4^{\text{SU}_3} &= H_1 - \frac{1}{2} L_{10} + O(1), \\ H_5^{\text{SU}_3} &= \frac{1}{9} \left(L_1 + \frac{1}{2} L_2 + \frac{1}{3} L_3 \right) + O(1), \\ H_6^{\text{SU}_3} &= F^4 (1 + A_1)^2 (72\tau)^{-1} + O(1). \end{aligned} \quad (96)$$

The coupling constants $L_{16}^{\text{SU}_3}$ and $H_6^{\text{SU}_3}$ receive a contribution from η' exchange, similar to the one in $L_7^{\text{SU}_3}$ (Sect. 18). The leading contribution to $L_{17}^{\text{SU}_3}$ stems from the Wess–Zumino–Witten term of the extended theory (Appendix C).

Note that, in the SU(3) framework, the Kaplan–Manohar transformation (47) takes the mass term of the effective Lagrangian into

$$\langle \bar{U}^\dagger \chi \rangle \rightarrow \langle \bar{U}^\dagger \chi \rangle + \frac{\lambda}{4B} \{ \langle \chi^\dagger \bar{U} \rangle^2 - \langle \chi^\dagger \bar{U} \chi^\dagger \bar{U} \rangle \},$$

even if the vacuum angle does not vanish: The factor $e^{-i\theta}$ is absorbed in the field \bar{U} , on account of $\det \bar{U} = e^{-i\theta}$. Hence the transformation of the quark mass matrix in (47) is equivalent to a change of the effective coupling constants occurring in $\mathcal{L}_{p^4}^{\text{SU}_3}$, also for $\theta \neq 0$.

C Renormalization of the WZW term

We first observe that the field \bar{U} introduced in (44) is renormalization group invariant. Moreover, U and \bar{U} transform in the same way under chiral rotations, because the factor that makes the difference, $e^{(i/3)\bar{\psi}}$, is invariant. This immediately implies that the anomalies of the functional S_{WZW} remain the same if U is replaced by \bar{U} . Indeed, performing the above change of variables in (80), we obtain a gauge invariant result for the difference:

$$\begin{aligned} S_{\text{WZW}}\{U, v, a\} &= S_{\text{WZW}}\{\bar{U}, v, a\} + \int A, \\ A &= -\frac{N_c}{144\pi^2} \bar{\psi} \langle iF_{\text{R}} D U D U^\dagger + F_{\text{R}} U F_{\text{L}} U^\dagger + 2F_{\text{R}}^2 \\ &\quad + (\text{R} \leftrightarrow \text{L}) \rangle, \\ F_{\text{R}} &= dr - ir^2, \quad F_{\text{L}} = dl - il^2, \end{aligned}$$

with $DU = dU - irU + iUl$. The terms occurring here are of the same form as those entering the $U(3)_{\text{R}} \times U(3)_{\text{L}}$ invariant part of the effective Lagrangian.

Next, we decompose the axial field according to (75). The corresponding decomposition of the right- and left-handed gauge fields into a renormalization group invariant part \bar{r}, \bar{l} and a remainder reads $r = \bar{r} + (1/6)D\theta$, $l = \bar{l} - (1/6)D\theta$. Using the identity $\langle d\bar{U}\bar{U}^\dagger \rangle = -id\theta$, which follows from $\det \bar{U} = e^{-i\theta}$, we then obtain

$$\begin{aligned} S_{\text{WZW}}\{U, v, a\} &= S_{\text{WZW}}\{\bar{U}, v, \bar{a}\} \\ &\quad + \int (A + B + P_1 + P_2), \end{aligned}$$

$$B = \frac{N_c}{144\pi^2} iD\theta \langle \bar{F}_R D\bar{U} D\bar{U}^\dagger - \bar{F}_L \bar{U}^\dagger D\bar{U} \rangle,$$

$$\bar{F}_R = F_R - \frac{1}{3} \langle F_R \rangle, \quad \bar{F}_L = F_L - \frac{1}{3} \langle F_L \rangle.$$

Note that terms proportional to $D\theta \langle (D\bar{U}\bar{U}^\dagger)^3 \rangle$ cancel out on account of charge conjugation invariance. The term $S_{\text{WZW}}\{\bar{U}, v, \bar{a}\}$ is renormalization group invariant. Under chiral rotations, it transforms with

$$\delta S_{\text{WZW}}\{\bar{U}, v, \bar{a}\} = \int \langle (\alpha_L - \alpha_R) \Omega_0 \rangle.$$

The one with B is gauge invariant, but transforms with Z_A^{-1} under the renormalization group. The calculation automatically yields the contact terms P_1, P_2 introduced in Sect. 19, which account for the difference between Ω and Ω_0 – these transform in a nontrivial manner, both under the renormalization group and under chiral rotations. Finally, A may also be sorted out according to the behavior under the renormalization group:

$$A = -\frac{N_c}{144\pi^2} \bar{\psi} \{ \langle i\bar{F}_R D\bar{U} D\bar{U}^\dagger + \bar{F}_R \bar{U} \bar{F}_L \bar{U}^\dagger + 2\bar{F}_R^2 \rangle + (\text{R} \leftrightarrow \text{L}) \} + \frac{1}{2} \langle F_R + F_L \rangle^2 + \frac{1}{6} \langle F_R - F_L \rangle^2.$$

This completes the decomposition of $S_{\text{WZW}}\{U, v, a\}$. The renormalization group invariant part is the Wess–Zumino–Witten term relevant for the effective theory at fixed N_c :

$$\int dx \mathcal{L}_{\text{WZW}}^{\text{SU}_3} \equiv S_{\text{WZW}}\{\bar{U}, v, \bar{a}\}. \quad (97)$$

The extension to the degrees of freedom carried by the η' contains the following additional contributions, which are gauge invariant:

$$\begin{aligned} \mathcal{L}_{\text{WZW}} = & \mathcal{L}_{\text{WZW}}^{\text{SU}_3} - \frac{N_c \epsilon^{\mu\nu\rho\sigma}}{288\pi^2} \{ \bar{\psi} \langle i\bar{R}_{\mu\nu} D_\rho \bar{U} D_\sigma \bar{U}^\dagger \\ & + i\bar{L}_{\mu\nu} D_\rho \bar{U}^\dagger D_\sigma \bar{U} + \bar{R}_{\mu\nu} \bar{U} \bar{L}_{\rho\sigma} \bar{U}^\dagger \rangle \\ & + \bar{\psi} \langle \bar{R}_{\mu\nu} \bar{R}_{\rho\sigma} + \bar{L}_{\mu\nu} \bar{L}_{\rho\sigma} \rangle \\ & - iD_\mu \theta \langle \bar{R}_{\nu\rho} D_\sigma \bar{U} \bar{U}^\dagger - \bar{L}_{\nu\rho} D_\sigma \bar{U}^\dagger \bar{U} \rangle \\ & + \frac{1}{4} \bar{\psi} \langle R_{\mu\nu} + L_{\mu\nu} \rangle \langle R_{\rho\sigma} + L_{\rho\sigma} \rangle \\ & + \left. \frac{1}{12} \bar{\psi} \langle R_{\mu\nu} - L_{\mu\nu} \rangle \langle R_{\rho\sigma} - L_{\rho\sigma} \rangle \right\}. \quad (98) \end{aligned}$$

References

1. G. 't Hooft, Nucl. Phys. B **72**, 461 (1974); *ibid.* B **75**, 461 (1974); G. Veneziano, Nucl. Phys. B **117**, 519 (1976); *ibid.* B **159**, 213 (1979); R.J. Crewther, Phys. Lett. B **70**, 349 (1977) and in *Field Theoretical Methods in Particle Physics*, edited by W. Rühl (Plenum, New York 1980); E. Witten, Nucl. Phys. B **156**, 269 (1979); *ibid.* B **160**, 57 (1979); P. Di Vecchia, Phys. Lett. B **85**, 357 (1979); P. Nath, R. Arnowitt, Phys. Rev. D **23**, 473 (1981)
2. S. Coleman, E. Witten, Phys. Rev. Lett. **45**, 100 (1980)
3. S. Weinberg, Physica A **96**, 327 (1979)
4. J. Gasser, H. Leutwyler, Nucl. Phys. B **250**, 465 (1985)
5. P. Di Vecchia, G. Veneziano, Nucl. Phys. B **171**, 253 (1980); C. Rosenzweig, J. Schechter, G. Trahern, Phys. Rev. D **21**, 3388 (1980); E. Witten, Ann. Phys. (N.Y.) **128**, 363 (1980); K. Kawarabayashi, N. Ohta, Nucl. Phys. B **175**, 477 (1980); P. Di Vecchia, F. Nicodemi, R. Pettorino, G. Veneziano, Nucl. Phys. B **181**, 318 (1981)
6. G.A. Christos, Phys. Rept. **116**, 251 (1984); Austral. J. Phys. **37**, 241 (1984)
7. N.I. Karchev, A.A. Slavnov, Theory. Math. Phys. **65**, 1099 (1985)
8. G. Grunberg, Nucl. Phys. B **269**, 318 (1986); Phys. Lett. B **168**, 141 (1986)
9. P. Minkowski, Phys. Lett. B **237**, 531 (1990)
10. J.M. Gérard, Mod. Phys. Lett. A **5**, 391 (1990)
11. J. Schechter, A. Subbaraman, H. Weigel, Phys. Rev. D **48**, 339 (1993) [hep-ph/9211239]
12. H. Georgi, Phys. Rev. D **49**, 1666 (1994) [hep-ph/9310337]
13. S. Peris, Phys. Lett. B **324**, 442 (1994) [hep-ph/9312239]; S. Peris, E. de Rafael, Phys. Lett. B **348**, 539 (1995) [hep-ph/9412343]; S. Peris, M. Perrottet, E. de Rafael, JHEP **9805**, 011 (1998) [hep-ph/9805442]
14. H. Leutwyler, Phys. Lett. B **374**, 163 (1996) [hep-ph/9601234]
15. P. Herrera-Siklody, J.I. Latorre, P. Pascual, J. Taron, Nucl. Phys. B **497**, 345 (1997) [hep-ph/9610549]
16. R. Kaiser, Diploma work, University of Bern (1997)
17. H. Leutwyler, in *Proceedings QCD 97*, Montpellier, France, July 1997, edited by S. Narison, Nucl. Phys. Proc. Suppl. **64**, 223 (1998) [hep-ph/9709408]
18. P. Herrera-Siklody, J.I. Latorre, P. Pascual, J. Taron, Phys. Lett. B **419**, 326 (1998) [hep-ph/9710268]
19. M. Knecht, E. de Rafael, Phys. Lett. B **424**, 335 (1998) [hep-ph/9712457]
20. R. Kaiser, H. Leutwyler, in *Proceedings Workshop Nonperturbative Methods in Quantum Field Theory*, NITP/CSSM, University of Adelaide, Australia, February 1998, eds. A.W. Schreiber, A.G. Williams, A.W. Thomas (World Scientific, Singapore, 1998) [hep-ph/9806336]
21. E. Witten, Phys. Rev. Lett. **81**, 2862 (1998) [hep-th/9807109]
22. P. Herrera-Siklody, Phys. Lett. B **442**, 359 (1998) [hep-ph/9808218]
23. T. Feldmann, Int. J. Mod. Phys. A **15**, 159 (2000) [hep-ph/9907491]
24. J. Dai, R. Dashen, E. Jenkins, A.V. Manohar, Phys. Rev. D **53**, 273 (1996) [hep-ph/9506273]; E. Jenkins, Ann. Rev. Nucl. Part. Sci. **48**, 81 (1998) [hep-ph/9803349]; Y. Oh, W. Weise, Eur. Phys. J. A **4**, 363 (1999) [hep-ph/9901354]; R. Flores-Mendieta, C.P. Hofmann, E. Jenkins, Phys. Rev. D **61**, 116014 (2000) [hep-ph/9911432]; R. Flores-Mendieta, C.P. Hofmann, E. Jenkins, A.V. Manohar, Phys. Rev. D **62**, 034001 (2000) [hep-ph/0001218]
25. R. Kaiser, H. Leutwyler, in preparation
26. P. Di Vecchia, G. Veneziano, Nucl. Phys. B **171**, 253 (1980); S. Narison, G.M. Shore, G. Veneziano, Nucl. Phys. B **433**, 209 (1995) [hep-ph/9404277]; *ibid.* B **546**, 235 (1999) [hep-ph/9812333]; G. Grunberg, Phys. Lett. B **388**, 173 (1996) [hep-ph/9607204]; B.L. Ioffe, A.G. Oganesian, Phys. Rev. D **57**, 6590 (1998) [hep-ph/9801345]; Acta Phys. Polon. B **29**, 2449 (1998); B.L. Ioffe, hep-ph/9811217
27. H. Leutwyler, A. Smilga, Phys. Rev. D **46**, 5607 (1992)

28. C. Vafa, E. Witten, Nucl. Phys. B **234**, 173 (1984)
29. D.B. Kaplan, A.V. Manohar, Phys. Rev. Lett. **56**, 2004 (1986)
30. S.L. Adler, Phys. Rev. **177**, 2426 (1969); J. Kodaira, Nucl. Phys. B **165**, 129 (1980); D. Espriu, R. Tarrach, Z. Phys. C **16**, 77 (1982)
31. H.W. Fearing, S. Scherer, Phys. Rev. D **53**, 315 (1996) [hep-ph/9408346]; J. Bijnens, G. Colangelo, G. Ecker, JHEP **9902**, 020 (1999) [hep-ph/9902437]
32. W.A. Bardeen, Phys. Rev. **184**, 1848 (1969)
33. J. Wess, B. Zumino, Phys. Lett. B **37**, 95 (1971); E. Witten, Nucl. Phys. B **223**, 422 (1983); K.C. Chou, H.Y. Guo, K. Wu, X.C. Song, Phys. Lett. B **134**, 67 (1984); Ö. Kaymakçalan, S. Rajeev, J. Schechter, Phys. Rev. D **30**, 594 (1984); H. Kawai, S.H.H. Tye, Phys. Lett. B **140**, 403 (1984); J.L. Mañes, Nucl. Phys. B **250**, 369 (1985); J. Bijnens, G. Ecker, J. Gasser, in The Second DAΦNE Physics Handbook, eds. L. Maiani, G. Pancheri, N. Paver (INFN-LNF-Divisione Ricerca, SIS-Ufficio Pubblicazioni, Frascati 1995) [hep-ph/9411232]
34. J.F. Donoghue, B.R. Holstein, Y.C.R. Lin, Phys. Rev. Lett. **55**, 2766 (1985); J. Bijnens, A. Bramon, F. Cornet, Phys. Rev. Lett. **61**, 1453 (1988); Z. Phys. C **46**, 599 (1990); J.F. Donoghue, D. Wyler, Nucl. Phys. B **316**, 289 (1989); D. Issler, in Proceedings 4th Meeting of the Division of Particles and Fields of the APS, Storrs, Connecticut, August 1988, edited by K. Haller et al. (World Scientific, 1988); B. Moussallam, Phys. Rev. D **51**, 4939 (1995) [hep-ph/9407402]
35. F.C. Hansen, H. Leutwyler, Nucl. Phys. B **350**, 201 (1991)